Limited-information strategies in Banach-Mazur games

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Toronto Set Theory Seminar
June 26, 2020
The Banach-Mazur game

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**EMPTY**

**NONEMPTY**
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EMPTY $U_0$

NONEMPTY

\[
U_0 \supseteq U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \ldots
\]

The first player (EMPTY) wins if $\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n = \emptyset$. Otherwise the second player (NONEMPTY) wins.
The Banach-Mazur game

In the Banach-Mazur game on a space $X$, two players take turns choosing members of an infinite sequence of nonempty open sets.

$$\text{EMPTY} \quad U_0 \subseteq \bigcup \quad \text{NONEMPTY} \quad V_0$$
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If $X$ is a compact Hausdorff space, then $\text{NONEMPTY}$ has a winning strategy in $\text{BM}(X)$:
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If \( X \) is a compact Hausdorff space, then \( \text{NONEMPTY} \) has a winning strategy in \( \text{BM}(X) \):

In the \( n^{th} \) round of the game, choose any nonempty open set \( V_n \) such that \( \overline{V_n} \subseteq U_n \).
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If $X$ is a compact Hausdorff space, then **NONEMPTY** has a winning strategy in BM($X$):
In the $n^{th}$ round of the game, choose any nonempty open set $V_n$ such that $V_n \subseteq U_n$.
Then $\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n \neq \emptyset$. 
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Given $k \in \omega$, a *winning $k$-tactic* is a winning strategy that depends only on the opponent’s previous $k$ moves. For example, the strategy for **NONEMPTY** in the second example on the previous slide is a winning 1-tactic.
Debs’ space

Theorem (Debs; 1985)

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Roughly, the proof that there is a 2-tactic uses topological features of the space to set up a coding mechanism, by which NONEMPTY is able to record, in each consecutive pair of her opponent’s moves, the entire history of the game up to that point.
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Telgársky’s conjecture

Conjecture (Telgársky; 1987)

For every $k \geq 2$, there is a topological space $X$ for which \textsc{Nonempty} has a winning $(k + 1)$-tactic, but no winning $k$-tactic.
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For every \( k \geq 2 \), there is a topological space \( X \) for which NONEMPTY has a winning \((k + 1)\)-tactic, but no winning \( k \)-tactic.

If true, Telgársky’s conjecture would imply:

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Proof: For each $k$, let $X_k$ be a space for which NONEMPTY has a
winning $(k + 1)$-tactic, but no winning $k$-tactic. Let $X = \bigcup_{k \in \mathbb{N}} X_k$. 

\[ X_1 \quad X_2 \quad X_3 \quad X_4 \quad X_5 \quad \ldots \]
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\textsc{Nonempty} has a winning strategy: After \textsc{Empty} plays $U_0 \subseteq X$, \textsc{Nonempty} may choose $V_0 \subseteq U_0 \cap X_k$ for some particular $k$, and then play with a winning strategy on $X_k$. 
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\[ \ldots \quad X_{k-1} \quad \bigcirc \quad X_k \quad X_{k+1} \quad X_{k+2} \quad \ldots \]

\( \text{NONEMPTY} \) has no winning \( k \)-tactic: If \( \text{NONEMPTY} \) had a winning \( k \)-tactic for \( X \), she would also have a winning \( k \)-tactic for \( X_k \), because \( \text{EMPTY} \) can play \( U_0 \subseteq X_k \).
Telgársky’s conjecture may fail

Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)

Assume GCH + □. For every $T_3$ space $X$, if NONEMPTY has a winning strategy, then she has a winning 2-tactic.

In particular, GCH + □ implies the failure of Telgársky’s conjecture for $T_3$ spaces.
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In particular, GCH + □ implies the failure of Telgársky’s conjecture for $T_3$ spaces (or, a little more generally, for quasi-regular spaces).

Roughly, the proof of this theorem shows that when GCH + □ holds, it is always possible to set up a coding mechanism (much like with Debs’ space, although this version is due to Fred Galvin) by which NONEMPTY is able to record, in each consecutive pair of her opponent’s moves, the entire history of the game up to that point.
Coding strategies

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Let us consider a Hausdorff space $X$ with a countable basis $\mathcal{B}$. Before talking about coding, we need an observation: NONEMPTY may always pretend, without loss of generality, that all of EMPTY's plays are elements of $\mathcal{B}$. If EMPTY plays some nonempty open set $U_n$, choose some $U'_n \in \mathcal{B}$ with $U'_n \subseteq U_n$. Then NONEMPTY may continue the game as if EMPTY had played $U'_n$ instead of $U_n$. If NONEMPTY has a winning strategy in the original game, that strategy still works with this alteration.
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![Diagram showing two nested open sets $U_n$ and $U'_n$.]

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Now suppose NONEMPTY can see only the last two moves of EMPTY. From this she can determine both what her countable sequence of sets was, and which one she pretended EMPTY played. From this, NONEMPTY can reconstruct the rest of the game’s history.
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Which spaces admit such a coding

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The countability of $\mathcal{B}$. Or more precisely, the fact that for every $U \in \mathcal{B}$, there is a surjection from some collection of disjoint open subsets of $U$ onto $\mathcal{B}^{<\omega}$.
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The countability of $B$. Or more precisely, the fact that for every $U \in B$, there is a surjection from some collection of disjoint open subsets of $U$ onto $B^{<\omega}$. In general, what we need is:

$\nabla(X)$: There is a pseudo-basis $B$ for $X$ such that for every $U \in B$, there is a collection $S$ of disjoint nonempty open subsets of $U$ such that $|\{V \in B : U \subseteq V\}| \leq |S|$. 

If this statement is true for some space $X$, then, via coding, $\text{NONEMPTY}$ has a winning strategy in $BM(X)$, then she has a winning 2-tactic. In particular, spaces satisfying this statement cannot witness Telgársky's conjecture.

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The countability of $\mathcal{B}$. Or more precisely, the fact that for every $U \in \mathcal{B}$, there is a surjection from some collection of disjoint open subsets of $U$ onto $\mathcal{B}^{<\omega}$. In general, what we need is:

$$\bigtriangledown(X): \text{There is a pseudo-basis } \mathcal{B} \text{ for } X \text{ such that for every } U \in \mathcal{B},$$
$$\text{there is a collection } S \text{ of disjoint nonempty open subsets of } U$$
$$\text{such that } |\{ V \in \mathcal{B} : U \subseteq V \}| \leq |S|.$$

If this statement is true for some space $X$, then, via coding,

*If NONEMPTY has a winning strategy in BM($X$), then she has a winning 2-tactic.*
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What topological properties of the space $X$ enabled us to set up this coding mechanism?

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\[ \nabla(X) \]: There is a pseudo-basis $B$ for $X$ such that for every $U \in B$, there is a collection $S$ of disjoint nonempty open subsets of $U$ such that $|\{ V \in B : U \subseteq V \}| \leq |S|$.

If this statement is true for some space $X$, then, via coding, \textit{If NONEMPTY has a winning strategy in BM($X$), then she has a winning 2-tactic.} In particular, spaces satisfying this statement cannot witness Telgársky’s conjecture.
Which spaces admit such a coding

It turns out that $\triangledown$ holding for $T_3$ spaces is really more of an order-theoretic proposition than a topological one:
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It turns out that $\nabla$ holding for $T_3$ spaces is really more of an order-theoretic proposition than a topological one:

**Theorem**

The following are equivalent:

- $\nabla(X)$ holds for every $T_3$ (or quasi-regular) space $X$. 

There are $T_2$ spaces $X$ that fail to satisfy $\nabla(X)$. But we do not know if such spaces can witness Telgársky’s conjecture.
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The following are equivalent:

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There are $T_2$ spaces $X$ that fail to satisfy $\nabla(X)$. But we do not know if such spaces can witness Telgársky’s conjecture.
Recall the main theorem under discussion:

**Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)**

*Assume GCH + □. For every $T_3$ space $X$, if NONEMPTY has a winning strategy, then she has a winning 2-tactic. In particular, GCH + □ implies the failure of Telgársky’s conjecture for $T_3$ spaces.*
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So far we have focused on the second implication, which is proved by the coding argument outlined above.
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Assume $\text{GCH} + \Box$. For every $T_3$ space $X$, if $\text{NONEMPTY}$ has a winning strategy, then she has a winning 2-tactic. In particular, $\text{GCH} + \Box$ implies the failure of Telgársky’s conjecture for $T_3$ spaces.

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So far we have focused on the second implication, which is proved by the coding argument outlined above.

What about the first implication?
A special case of $\nabla$ provable from ZFC

First let’s consider a special case of $\nabla$ that can be proved from ZFC: Suppose $\mathbb{P}$ is a separative poset with $|\mathbb{P}| = \aleph_1$. Enumerate $\mathbb{P} = \{ p_\alpha : \alpha < \omega_1 \}$, and define $D = \{ p_\alpha : \text{if } \beta < \alpha \text{ then } p_\beta \text{ is not an extension of } p_\alpha \}$. $D$ is dense in $\mathbb{P}$, because for any given $p_\alpha \in \mathbb{P}$, if $\beta = \min \{ \xi < \omega_1 : p_\xi \text{ extends } p_\alpha \}$, then $p_\beta \in D$. For any given $p_\alpha \in \mathbb{P}$, our definition of $D$ ensures that $\{ d \in D : p_\alpha \text{ extends } d \} \subseteq \{ p_\beta : \beta \leq \alpha \}$, and therefore $\{ d \in D : p_\alpha \text{ extends } d \}$ is countable. It follows that if $\mathbb{P}$ has the $\aleph_1$-cc (i.e., $\mathbb{P}$ is ccc), then this dense set $D$ witnesses that $\nabla$ holds for $\mathbb{P}$. (If $\mathbb{P}$ does not have the $\aleph_1$-cc, then $\nabla$ holds for $\mathbb{P}$ trivially, by taking $D = \mathbb{P}$.)

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GCH + □ and magic enumerations of posets

Roughly, GCH + □ enables us to use a similar argument for larger \( \mathbb{P} \).
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In general, any enumeration of a ccc poset \( \mathbb{P} \) gives rise to a dense \( \mathbb{D} \subseteq \mathbb{P} \) via a greedy algorithm, as on the previous slide.
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In general, any enumeration of a ccc poset $\mathbb{P}$ gives rise to a dense $D \subseteq \mathbb{P}$ via a greedy algorithm, as on the previous slide. But when $|\mathbb{P}| > \aleph_1$, the dense set given by an arbitrary enumeration of $\mathbb{P}$ may no longer witness $\nabla$. 
Roughly, GCH + □ enables us to use a similar argument for larger $P$. In general, any enumeration of a ccc poset $P$ gives rise to a dense $D \subseteq P$ via a greedy algorithm, as on the previous slide. But when $|P| > \aleph_1$, the dense set given by an arbitrary enumeration of $P$ may no longer witness $\nabla$. The reason is that while upward cones in $D$ are still contained in initial segments of our enumeration, these initial segments are not always countable.
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Special enumerations of $\mathbb{P}$ are needed to make the argument work.
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In general, any enumeration of a ccc poset \( P \) gives rise to a dense \( D \subseteq P \) via a greedy algorithm, as on the previous slide. But when \( |P| > \aleph_1 \), the dense set given by an arbitrary enumeration of \( P \) may no longer witness \( \nabla \). The reason is that while upward cones in \( D \) are still contained in initial segments of our enumeration, these initial segments are not always countable.

Special enumerations of \( P \) are needed to make the argument work.

These enumerations arise from special chains of elementary submodels called *high Davies trees*, which are constructed via GCH + □.
A *high Davies tree* for $\mathbb{P}$ over $\mu$ is a sequence $\langle M_\alpha : \alpha < \mu \rangle$ of elementary submodels of some "sufficiently large" fragment $H$ of the set-theoretic universe such that

- $\mathbb{P} \in M_\alpha$,
- $M_\alpha$ is countably closed,
- $|M_\alpha| = \aleph_1$ for every $\alpha$,
- $\mathbb{P} \subseteq \bigcup_{\alpha < \mu} M_\alpha$,
- and for each $\alpha < \mu$, there is a countable set $N_\alpha$ of countably closed elementary submodels of $H$, each containing $\mathbb{P}$, with $\bigcup_{\xi < \alpha} M_\xi = \bigcup N_\alpha$.
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Theorem (Soukup and Soukup; 2017)

Assuming $\text{GCH} + \square$, if $\mathbb{P}$ is any set and $\mu$ any regular uncountable cardinal with $\mu \geq |\mathbb{P}|$, then there is a high Davies tree for $\mathbb{P}$ over $\mu$. 

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Limited-information strategies in Banach-Mazur games
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High Davies trees are so named because they can be constructed by enumerating the leaves of a tree of elementary submodels.

\[ M_{\alpha,0} \prec M_{\alpha,1} \prec \cdots \prec M_{\alpha,\beta} \prec M_{\alpha,\beta+1} \prec \cdots \]

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High Davies trees can be used to prove that $\nabla$ holds for ccc posets. For the general case of $\kappa$-cc posets, we needed a version with stronger closure properties, called $\kappa$-high Davies trees. But in what follows, we restrict our attention to the ccc case.
Proof sketch: from high Davies trees to $\nabla$

Let $\mathbb{P}$ be a separative ccc poset, and suppose $\langle M_\alpha : \alpha < \mu \rangle$ is a high Davies tree for $\mathbb{P}$ over some $\mu \geq |\mathbb{P}|$. 

For each $\alpha < \kappa$, recall that $|M_\alpha| = \aleph_1$ and fix a well ordering $\preceq_\alpha$ of $M_\alpha$ with order type $\omega_1$. Then define a well ordering of $\mathbb{P}$ as follows:

- If $p$ appears in an earlier part of the Davies tree than $q$ does, by which we mean that there is some $\alpha$ with $p \in M_\alpha$ but $q \not\in \bigcup \xi \leq \alpha M_\xi$, then we define $p \preceq q$.
- Similarly, if $q$ appears earlier than $p$ then $q \preceq p$.
- Otherwise, there is some (unique) $\alpha < \mu$ such that $p, q \in M_\alpha \setminus \bigcup \xi < \alpha M_\xi$. In this case we define $p \preceq q$ if and only if $p \preceq_\alpha q$.

This is the promised “special enumeration” of $\mathbb{P}$ that will make our greedy algorithm work.
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Limited-information strategies in Banach-Mazur games
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Let $\mathbb{P}$ be a separative ccc poset, and suppose $\langle M_\alpha : \alpha < \mu \rangle$ is a
high Davies tree for $\mathbb{P}$ over some $\mu \geq |\mathbb{P}|$.

For each $\alpha < \kappa$, recall that $|M_\alpha| = \aleph_1$ and fix a well ordering $\sqsupseteq_\alpha$ of
$M_\alpha$ with order type $\omega_1$. Then define a well ordering of $\mathbb{P}$ as follows:
for every $p, q \in \mathbb{P}$,

- If $p$ appears in an earlier part of the Davies tree than $q$ does,
  by which we mean that there is some $\alpha$ with $p \in M_\alpha$ but
  $q \notin \bigcup_{\xi \leq \alpha} M_\xi$, then we define $p \sqsupseteq q$.
- Similarly, if $q$ appears earlier than $p$ then $q \sqsubseteq p$.
- Otherwise, there is some (unique) $\alpha < \mu$ such that
  $p, q \in M_\alpha \setminus \bigcup_{\xi < \alpha} M_\xi$. In this case we define $p \sqsupseteq q$ if and
  only if $p \sqsupseteq_\alpha q$.

This is the promised "special enumeration" of $\mathbb{P}$ that will make our
greedy algorithm work.
Proof sketch: from high Davies trees to $\nabla$

As before, define

$$D = \{ q \in \mathbb{P} : \text{if } p \sqsubseteq q \text{ then } p \text{ is not an extension of } q \}.$$
Proof sketch: from high Davies trees to ▼

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\[ \mathcal{D} = \{ q \in \mathcal{P} : \text{if } p \sqsubset q \text{ then } p \text{ is not an extension of } q \} \].

• \( \mathcal{D} \) is dense in \( \mathcal{P} \), because for any given \( p \in \mathcal{P} \), the \( \sqsubseteq \)-least element of \( \{ q \in \mathcal{P} : q \text{ extends } p \} \) must be in \( \mathcal{D} \).
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- $\mathbb{D}$ is dense in $\mathbb{P}$, because for any given $p \in \mathbb{P}$, the $\sqsubseteq$-least element of $\{ q \in \mathbb{P} : q \text{ extends } p \}$ must be in $\mathbb{D}$.
- For any given $p \in \mathbb{P}$, our definition of $\mathbb{D}$ ensures that $\{ d \in \mathbb{D} : p \text{ extends } d \} \subseteq \{ d \in \mathbb{P} : d \sqsubseteq p \}$.
Proof sketch: from high Davies trees to $\nabla$

As before, define

$$D = \{ q \in P : \text{if } p \sqsubset q \text{ then } p \text{ is not an extension of } q \}.$$ 

- $D$ is dense in $P$, because for any given $p \in P$, the $\sqsubseteq$-least element of $\{ q \in P : q \text{ extends } p \}$ must be in $D$.
- For any given $p \in P$, our definition of $D$ ensures that $\{ d \in D : p \text{ extends } d \} \subseteq \{ d \in P : d \sqsubseteq p \}$.

To prove that $\nabla$ holds for $P$, we need to show that for any $p \in P$, $\{ d \in D : p \text{ extends } d \}$ is countable.
**Proof sketch: from high Davies trees to ▽**

As before, define

\[ \mathbb{D} = \{q \in \mathbb{P} : \text{if } p \sqsubseteq q \text{ then } p \text{ is not an extension of } q\} \].

- \( \mathbb{D} \) is dense in \( \mathbb{P} \), because for any given \( p \in \mathbb{P} \), the \( \sqsubseteq \)-least element of \( \{q \in \mathbb{P} : q \text{ extends } p\} \) must be in \( \mathbb{D} \).
- For any given \( p \in \mathbb{P} \), our definition of \( \mathbb{D} \) ensures that \( \{d \in \mathbb{D} : p \text{ extends } d\} \subseteq \{d \in \mathbb{P} : d \sqsubseteq p\} \).

To prove that ▽ holds for \( \mathbb{P} \), we need to show that for any \( p \in \mathbb{P} \), \( \{d \in \mathbb{D} : p \text{ extends } d\} \) is countable.

Aiming for a contradiction, let us suppose \( \{d \in \mathbb{D} : p \text{ extends } d\} \) is uncountable; furthermore, let us suppose that \( p \) is the \( \sqsubseteq \)-least element of \( \mathbb{P} \) with this property.
Proof sketch: from high Davies trees to ▽

As before, define

\[ D = \{ q \in P : \text{if } p \sqsubseteq q \text{ then } p \text{ is not an extension of } q \} \].

- \( D \) is dense in \( P \), because for any given \( p \in P \), the \( \sqsubseteq \)-least element of \( \{ q \in P : q \text{ extends } p \} \) must be in \( D \).
- For any given \( p \in P \), our definition of \( D \) ensures that \( \{ d \in D : p \text{ extends } d \} \subseteq \{ d \in P : d \sqsubseteq p \} \).

To prove that ▽ holds for \( P \), we need to show that for any \( p \in P \), \( \{ d \in D : p \text{ extends } d \} \) is countable.

Aiming for a contradiction, let us suppose \( \{ d \in D : p \text{ extends } d \} \) is uncountable; furthermore, let us suppose that \( p \) is the \( \sqsubseteq \)-least element of \( P \) with this property. Let \( \alpha \) denote the stage at which \( p \) appears in our Davies tree: i.e., \( p \in M_\alpha \setminus \bigcup_{\xi < \alpha} M_\xi \).
Proof sketch: from high Davies trees to $\bigtriangledown$

Let $Q = \{ q \in D : p \text{ extends } d \}$, and recall that every member of $Q$ is a $\sqsubseteq$-predecessor of $p$. 

Because $p$ has only countably many $\sqsubseteq$-predecessors in $M_\alpha$, $Q \cap \bigcup_{\xi < \alpha} M_\xi$ is uncountable.

By the definition of a high Davies tree, there is a countable set $N_\alpha$ of countably closed elementary submodels of $H$, each containing $P$, with $\bigcup_{\xi < \alpha} M_\xi = \bigcup N_\alpha$.

By the pigeonhole principle, there is some $N \in N_\alpha$ such that $N \cap Q \cap \bigcup_{\xi < \alpha} M_\xi$ is uncountable.

Because $N$ is a countably closed model of (enough of) ZFC and $P$ has the ccc, $N$ contains some $p' \in P$ that extends every member of the uncountable set $N \cap Q \cap \bigcup_{\xi < \alpha} M_\xi$.

But $Q \subseteq D$, so $p'$ extends uncountably many elements of $D$.

Because $p' \in N \subseteq \bigcup_{\xi < \alpha} M_\xi$, we also have $p' \sqsupseteq p$.

This contradicts our choice of $p$. 

Will Brian
Limited-information strategies in Banach-Mazur games
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Let $Q = \{ q \in D : p \text{ extends } d \}$, and recall that every member of $Q$ is a $\sqsubseteq$-predecessor of $p$. Because $p$ has only countably many $\sqsubseteq$-predecessors in $M_\alpha$, $Q \cap \bigcup_{\xi < \alpha} M_\xi$ is uncountable.
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Proof sketch: from high Davies trees to $\Diamond$

Let $Q = \{ q \in D : p \text{ extends } d \}$, and recall that every member of $Q$ is a $\Box$-predecessor of $p$. Because $p$ has only countably many $\Box$-predecessors in $M_\alpha$, $Q \cap \bigcup_{\xi < \alpha} M_\xi$ is uncountable.

By the definition of a high Davies tree, there is a countable set $N_\alpha$ of countably closed elementary submodels of $H$, each containing $\mathbb{P}$, with $\bigcup_{\xi < \alpha} M_\xi = \bigcup N_\alpha$. By the pigeonhole principle, there is some $N \in N_\alpha$ such that $N \cap Q \cap \bigcup_{\xi < \alpha} M_\xi$ is uncountable.

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Limited-information strategies in Banach-Mazur games
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This contradicts our choice of $p$. \qed
GCH and $\nabla$

Our proof only uses □ on singular cardinals. Hence

**Corollary**

*If $\mathbb{P}$ is a separative poset with $|\mathbb{P}| \leq \aleph_\omega$ and GCH holds below $|\mathbb{P}|$, then $\nabla$ holds for $\mathbb{P}$.***
Our proof only uses □ on singular cardinals. Hence

**Corollary**

*If \( P \) is a separative poset with \( |P| \leq \aleph_\omega \) and GCH holds below \( |P| \), then \( \nabla \) holds for \( P \).*

However, some form of □ on singular cardinals is necessary.

**Theorem (Brian, Dow, and Shelah; 2020)**

*Assuming the existence of a huge cardinal, there is a model satisfying GCH but not \( \nabla \). Therefore GCH does not imply \( \nabla \) (unless huge cardinals are inconsistent).*
GCH and \( \nabla \)

Our proof only uses \( \square \) on singular cardinals. Hence

**Corollary**

*If \( \mathbb{P} \) is a separative poset with \( |\mathbb{P}| \leq \aleph_\omega \) and GCH holds below \( |\mathbb{P}| \), then \( \nabla \) holds for \( \mathbb{P} \).*

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*Assuming the existence of a huge cardinal, there is a model satisfying GCH but not \( \nabla \). Therefore GCH does not imply \( \nabla \) (unless huge cardinals are inconsistent).*

To show that GCH does not imply \( \nabla \) requires getting a model of GCH+ the failure of \( \square \) on some singular cardinals. The existence of such a model requires fairly strong large cardinal axioms.
GCH and ▽

Theorem (Brian, Dow, and Shelah; 2020)

Assuming the existence of a huge cardinal, there is a model satisfying GCH but not ▽.

Proof idea: The proof uses a form of Chang’s conjecture known as Chang’s conjecture for $\aleph_\omega$, denoted $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. It is known that $\text{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ is consistent relative to a huge cardinal.
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We construct a ccc poset $P$ (a modified finite support product of Hechler forcings), and then use $(\aleph_\omega+1, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$, to show that $P$ fails to satisfy $\nabla$. 

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Theorem (Brian, Dow, and Shelah; 2020)

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We construct a ccc poset \( \mathbb{P} \) (a modified finite support product of Hechler forcings), and then use \((\aleph_{\omega+1}, \aleph_{\omega}) \rightarrow (\aleph_1, \aleph_0)\), to show that \( \mathbb{P} \) fails to satisfy ▽.

We also show that if we begin with a model of GCH + \((\aleph_{\omega+1}, \aleph_{\omega}) \rightarrow (\aleph_1, \aleph_0)\) and then force with a finite support product of \( \aleph_1 \) amoeba forcings, then in the extension GCH still holds but ▽ fails for the measure algebra of weight \( \aleph_{\omega} \).
If we do not insist on GCH, it is much easier to get $\nabla$ to fail.
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Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)

$\nabla$ implies $b = \aleph_1$. 

Proof idea: Using $b > \aleph_1$, a pigeonhole argument shows that the Hechler forcing fails to satisfy $\nabla$. A little more generally, $\nabla$ fails whenever there is a descending sequence in the poset $(\mathcal{P}(\omega)/\text{fin}, \subseteq \ast)$ with order type $\omega_2$. However, $\nabla$ does not imply GCH or even CH.
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A little more generally, $\nabla$ fails whenever there is a descending sequence in the poset $(\mathcal{P}(\omega)/\text{fin}, \subseteq^*)$ with order type $\omega_2$. 
Telgársky’s Conjecture
\( GCH, \square, \text{ and } \bigtriangleup \)

\( \bigtriangleup \) without CH

If we do not insist on GCH, it is much easier to get \( \bigtriangleup \) to fail.

**Theorem (Brian, Dow, Milovich, and Yengulalp; 2020)**

\( \bigtriangleup \) implies \( b = \aleph_1. \)

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A little more generally, \( \bigtriangleup \) fails whenever there is a descending sequence in the poset \( (\mathcal{P}(\omega)/\text{fin}, \subseteq^*) \) with order type \( \omega_2 \).

However, \( \bigtriangleup \) does not imply GCH or even CH.

**Theorem (Brian, Dow, and Shelah; 2020)**

*If GCH + \( \square \) holds, then \( \bigtriangleup \) still holds after forcing with \( \text{Fn}(\kappa, 2) \) to add \( \kappa \) Cohen reals (for any cardinal \( \kappa \)). Therefore \( \bigtriangleup \) does not imply CH.*
Open questions

Question

Is it consistent that there is a sequence of $T_3$ spaces witnessing Telgársky’s conjecture?
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Question

How badly can ▽ fail? Specifically, is it consistent to have a ccc poset $\mathbb{P}$ such that for every dense $\mathbb{D} \subseteq \mathbb{P}$, there is some $p \in \mathbb{P}$ with $|\{d \in \mathbb{D} : p \text{ extends } d\}| \geq \mathfrak{c}^+$? Can we get the size arbitrarily high above $\mathfrak{c}$?
Thank you for listening