Self-homeomorphisms of $\mathbb{N}^*$ and their quotients

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- Every compact Hausdorff space of weight \( \leq \aleph_1 \) is a continuous image of \( \mathbb{N}^* \) (Parovičenko, 1963).
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- $\mathbb{N}^*$ a compact Hausdorff space. It can be viewed as the space of all non-principal ultrafilters on $\mathbb{N}$.
- Every compact Hausdorff space of weight $\leq \aleph_1$ is a continuous image of $\mathbb{N}^*$ (Parovičenko, 1963).
- This is a kind of “universal property” for $\mathbb{N}^*$. Assuming the Continuum Hypothesis, it says that every compact Hausdorff space of weight $\leq \aleph_1$ is a continuous image of $\mathbb{N}^*$. 
trivial self-maps of $\mathbb{N}^*$

- A *mod-finite permutation* of $\mathbb{N}$ is a bijection $p : A \rightarrow B$, where $A, B$ are co-finite subsets of $\mathbb{N}$.
A mod-finite permutation of \( \mathbb{N} \) is a bijection \( p : A \rightarrow B \), where \( A, B \) are co-finite subsets of \( \mathbb{N} \).

Example: the successor function \( s(n) = n + 1 \)

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If $p$ is a mod-finite permutation of $\mathbb{N}$, then it induces a self-homeomorphism $p^* : \mathbb{N}^* \to \mathbb{N}^*$, defined by taking $A \in p^*(\mathcal{U}) \iff p^{-1}[A] \in \mathcal{U}$.
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Self-homeomorphisms of \( \mathbb{N}^* \) arising in this way are called *trivial*. 

**trivial self-maps of \( \mathbb{N}^* \)**
What are the quotients of these trivial maps?

Recall that a map $f : X \to X$ is a *quotient* of a map $g : Y \to Y$ if there is a continuous surjection $q : Y \to X$ such that $q \circ g = f \circ q$.

![Diagram](image_url)
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\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y \\
\downarrow q & & \downarrow q \\
X & \xrightarrow{f} & X
\end{array}
\]

\textbf{Question}

\textit{If $p$ is a mod-finite permutation of $\mathbb{N}$, then what are the quotients of $p^* : \mathbb{N}^* \to \mathbb{N}^*$?}
An external characterization

**Theorem**

\[ f : X \to X \text{ is a quotient of } p^* \text{ if and only if there is some } Z \supseteq X \text{ and some } h : Z \to Z \text{ such that, in } Z, X \text{ is the limit of a "}p\text{-like}\text{" sequence of points}. \]
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What about an “internal” characterization of these quotients?
Finite diagrams of a map

Suppose $f : X \to X$ is a continuous function and $\mathcal{V}$ is an open cover of $X$. 

Consider the relation on $\mathcal{V}$ given by $V \to W$ iff $f(V) \cap W \neq \emptyset$. The structure $(\mathcal{V}, \to)$ is a directed graph (possibly with loops). Any loopy directed graph that is isomorphic to one arising this way is called a diagram $D_f$. Roughly, a finite diagram encodes a finite amount of combinatorial information about the structure of $f$. 

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Roughly, a finite diagram encodes a finite amount of combinatorial information about the structure of $f$. 
Two examples

The map:

\[
\begin{align*}
\text{id} & \quad \text{on } [0, 1] \\
\theta & \mapsto \theta + \frac{\pi}{2}
\end{align*}
\]

on $S^1$
Two examples

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the cover:

the digraph:
Observation

If $f : X \to X$ is a quotient of $g : Y \to Y$, then every diagram for $f$ is also a diagram for $g$. 
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Proof:

- Let $f : X \to X$ be a quotient of $g : Y \to Y$, and let $q : Y \to X$ be a quotient mapping.
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Proof:

- Let $f : X \to X$ be a quotient of $g : Y \to Y$, and let $q : Y \to X$ be a quotient mapping.
- Let $\mathcal{V}$ be an open cover for $X$ witnessing that some particular digraph $G$ is a diagram for $f$. 
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- Let $\mathcal{V}$ be an open cover for $X$ witnessing that some particular digraph $G$ is a diagram for $f$.
- Pull $\mathcal{V}$ back under $q$ to get an open cover of $Y$. 
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Proof:

- Let \( f : X \to X \) be a quotient of \( g : Y \to Y \), and let \( q : Y \to X \) be a quotient mapping.
- Let \( \mathcal{V} \) be an open cover for \( X \) witnessing that some particular digraph \( G \) is a diagram for \( f \).
- Pull \( \mathcal{V} \) back under \( q \) to get an open cover of \( Y \).
- Because \( q \circ g = f \circ q \), this open cover will witness that \( G \) is a diagram for \( g \).
For trivial self-homeomorphisms of $\mathbb{N}^*$, this necessary condition turns out to be sufficient as well:

**Main Theorem**

Let $f : X \to X$ be continuous, with the weight of $X$ at most $\aleph_1$. If $p$ is a mod-finite permutation of $\mathbb{N}$, then $f$ is a quotient of $p^*$ if and only if every finite diagram for $f$ is also a diagram for $p^*$.
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### Main Theorem

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### Corollary

Let $p$ be a mod-finite permutation of $\mathbb{N}$. Assuming the Continuum Hypothesis, a continuous function $f : X \to X$ is a quotient of $p^*$ if and only if

1. $X$ is a continuous image of $\mathbb{N}^*$, and
2. every finite diagram for $f$ is also a diagram for $p^*$. 
Let \( t \) be the permutation of \( \mathbb{N} \times \mathbb{Z} \) given by \( t(n, z) = (n, z + 1) \).
(By reindexing, we may consider it a permutation of \( \mathbb{N} \)).

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**Corollary**

*Every dynamical system of weight $\leq \aleph_1$ is a quotient of $t^*$.*
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\vdots \quad \vdots \quad \vdots \quad \vdots
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**Corollary**

*Every dynamical system of weight $\leq \aleph_1$ is a quotient of $t^*$. Consequently, the Continuum Hypothesis implies that $t^*$ is a "universal" dynamical system of weight $\leq c$.***
By applying Stone duality, we may translate this result into a statement about Boolean algebras:

**Dual corollary**

Let \( \tau \) be the automorphism of the Boolean algebra \( \mathcal{P}(\mathbb{N})/\text{fin} \) induced by the map \( t \).
By applying Stone duality, we may translate this result into a statement about Boolean algebras:

**Dual corollary**

Let \( \tau \) be the automorphism of the Boolean algebra \( \mathcal{P}(\mathbb{N})/\text{fin} \) induced by the map \( t \).

If \( \alpha \) is any automorphism of any Boolean algebra \( A \) of size \( \leq \aleph_1 \), then there is a subalgebra \( B \) of \( \mathcal{P}(\mathbb{N})/\text{fin} \) such that \((A, \alpha)\) is isomorphic to \((B, \tau \restriction B)\).
By applying Stone duality, we may translate this result into a statement about Boolean algebras:

**Dual corollary**

Let $\tau$ be the automorphism of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{fin}$ induced by the map $t$.

If $\alpha$ is any automorphism of any Boolean algebra $\mathcal{A}$ of size $\leq \aleph_1$, then there is a subalgebra $\mathcal{B}$ of $\mathcal{P}(\mathbb{N})/\text{fin}$ such that $(\mathcal{A}, \alpha)$ is isomorphic to $(\mathcal{B}, \tau_{\restriction \mathcal{B}})$.

Consequently, the Continuum Hypothesis implies that $\tau$ is a “universal” automorphism for Boolean algebras of size $\leq c$. 
"full" dynamical systems

Let us recall our observation from a previous slide:

\[ f \text{ is a quotient of } g \implies \text{every finite diagram for } f \text{ is a diagram for } g. \]
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**Definition**

Let us say that a dynamical system \( g \) is \textit{full} if this implication reverses for all metrizable \( f \).

Roughly, \( g \) is full if it has as many metrizable quotients as possible.
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**Question**

*What dynamical systems have this property? Can we classify them, or at least prove interesting theorems in this direction?*
Thank you for listening