About $\beta N$

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Measures on the natural numbers

An *ultrafilter* is a function $\mu : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that

1. $\mu(\emptyset) = 0$ and $\mu(\mathbb{N}) = 1$.
2. If $A_0, A_1, \ldots, A_n$ are disjoint subsets of $\mathbb{N}$, then

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\mu \left( \bigcup_{i \leq n} A_i \right) = \sum_{i \leq n} \mu(A_i).
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In other words, an ultrafilter is a finitely additive 0-1-valued measure on the natural numbers, for which every subset of $\mathbb{N}$ is measurable.
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Observe that:

3. If $A_0, A_1, \ldots, A_n$ form a partition of $\mathbb{N}$, then $\mu$ assigns measure 1 to precisely one of the $A_i$, and measure 0 to all the rest.
Principal vs. non-principal

For example, to each $n \in \mathbb{N}$ we can associate an ultrafilter $\mu_n$, the “delta measure” at $n$, defined by

$$\mu_n(A) = \begin{cases} 0 & \text{if } n \notin A \\ 1 & \text{if } n \in A. \end{cases}$$
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- By finite additivity, an ultrafilter $\mu$ is non-principal if and only if $\mu(F) = 0$ for every finite $F \subseteq \mathbb{N}$.
- The existence of non-principal ultrafilters is proved using the Axiom of Choice.
Ultrafilters can be used to pick out limits of sequences:
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Our first definition of $\beta\mathbb{N}$

More generally, if $X$ is any compact Hausdorff space and \( \langle x_n : n \in \mathbb{N} \rangle \) is any sequence of points in $X$, then an ultrafilter $\mu$ can be used to assign a unique “limit” to the sequence:
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$\beta N$ is the set of all ultrafilters.
The space of all ultrafilters

- \(\beta\mathbb{N}\) has a standard topology on it. For every \(A \subseteq \mathbb{N}\), let

\[
\overline{A} = \{\mu \in \beta\mathbb{N} : \mu(A) = 1\}.
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The sets of this form provide a basis for the standard topology on \(\beta\mathbb{N}\). In this topology, \(\overline{A}\) is both open and closed.
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- With this topology, $\beta \mathbb{N}$ is a compact Hausdorff space.

- $\mathbb{N}^*$ denotes the subspace of $\beta \mathbb{N}$ consisting of only the non-principal ultrafilters. $\mathbb{N}^*$ is also a compact Hausdorff space.
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- The space $\beta\mathbb{N}$ is known as the *Stone-Čech compactification* of $\mathbb{N}$, and $\mathbb{N}^*$ is known as its *remainder*. 
It’s a strange space

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- $\beta\mathbb{N}$ is separable, but not hereditarily separable. $\mathbb{N}^*$ is a non-separable subspace of $\beta\mathbb{N}$. 
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- In fact, every separable subspace of $\mathbb{N}^*$ is nowhere dense.
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- Neither space is metrizable. In fact, neither space contains a convergent sequence.
- $\beta \mathbb{N}$ is separable, but not hereditarily separable. $\mathbb{N}^*$ is a non-separable subspace of $\beta \mathbb{N}$.
- In fact, every separable subspace of $\mathbb{N}^*$ is nowhere dense.
- Many topological properties of $\beta \mathbb{N}$ and $\mathbb{N}^*$ are known to be independent of ZFC.
\(\beta N\) is a set, a topological space, a dynamical system, and a semigroup. It’s a universal space.

\(\beta N\) is “universal” for sufficiently small compact Hausdorff spaces:

\[\text{Theorem}\]

Every compact metric space is a continuous image of \(\beta N\) and \(N^*\).

(Parovičenko, 1963)

Every compact space of weight \(\leq \aleph_1\) is a continuous image of \(N^\ast\).

(Kunen, 1968)

The same cannot necessarily be said for spaces of weight \(\aleph_2\), even when the Continuum Hypothesis fails badly.
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\( \beta \mathbb{N} \) is “universal” for sufficiently small compact Hausdorff spaces:

**Theorem**

- Every compact metric space is a continuous image of \( \beta \mathbb{N} \) and \( \mathbb{N}^* \).
- (Parovičenko, 1963) Every compact space of weight \( \leq \aleph_1 \) is a continuous image of \( \mathbb{N}^* \).
- (Kunen, 1968) The same cannot necessarily be said for spaces of weight \( \aleph_2 \), even when the Continuum Hypothesis fails badly.
A *dynamical system* is a compact Hausdorff space $X$ and a continuous self-map $f : X \to X$. 
Dynamical systems and omega-limit sets

A *dynamical system* is a compact Hausdorff space $X$ and a continuous self-map $f : X \rightarrow X$.

Given a dynamical system $(X, f)$ and a point $x \in X$, the *omega-limit set* of $x$ is the set of all limit points of the orbit of $x$:

$$\omega_f(x) = \bigcap_{n \in \mathbb{N}} \{ f^m(x) : m \geq n \}.$$
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[Diagram showing a point $x$ and its orbit $f(x)$ within the space $X$.]
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An omega-limit set

\[ x \xrightarrow{f} f(x) \xrightarrow{f^2} f^2(x) \xrightarrow{f^3} f^3(x) \xrightarrow{\cdots} \omega_f(x) \]
Abstract omega-limit sets

An *abstract omega-limit set* is a dynamical system that is isomorphic (or conjugate) to an omega-limit set.
There is a standard map $\sigma : \beta \mathbb{N} \to \beta \mathbb{N}$ making $\beta \mathbb{N}$ into a dynamical system: given $\mu \in \beta \mathbb{N}$, $\sigma(\mu)$ is defined to be the unique ultrafilter such that, for every $A \subseteq \mathbb{N}$,

$$\sigma(\mu)(A) = 1 \iff \mu(A - 1) = 1.$$
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This map is called the \textit{shift map}. It is a continuous injection \( \beta \mathbb{N} \rightarrow \beta \mathbb{N} \), and it restricts to a self-homeomorphism \( \mathbb{N}^* \rightarrow \mathbb{N}^* \).
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This map is called the \textit{shift map}. It is a continuous injection $\beta\mathbb{N} \to \beta\mathbb{N}$, and it restricts to a self-homeomorphism $\mathbb{N}^* \to \mathbb{N}^*$. Within $(\beta\mathbb{N}, \sigma)$, $\mathbb{N}^*$ is an omega limit set: if $\mu_n$ is any principal ultrafilter then $\omega_{\sigma}(\mu_n) = \mathbb{N}^*$. 
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Within $(\beta \mathbb{N}, \sigma)$, $\mathbb{N}^*$ is an omega limit set: if $\mu_n$ is any principal ultrafilter then $\omega_\sigma(\mu_n) = \mathbb{N}^*$. Therefore $(\mathbb{N}^*, \sigma)$ is an abstract omega-limit set.
Universality, again . . .

\((\mathbb{N}^*, \sigma)\) is universal as an abstract omega-limit set:
Universality, again . . .

$(\mathbb{N}^*, \sigma)$ is universal as an abstract omega-limit set:

**Theorem**

A dynamical system is an abstract omega-limit set if and only if it is a continuous image of $(\mathbb{N}^*, \sigma)$.
Universality, again . . .

$$(\mathbb{N}^*, \sigma)$$ is universal as an abstract omega-limit set:

**Theorem**

A dynamical system is an abstract omega-limit set if and only if it is a continuous image of $$(\mathbb{N}^*, \sigma)$$.

One direction of this theorem is proved by taking limits along ultrafilters: if $$(X, f)$$ is a dynamical system and $x \in X$, then $\mu \mapsto \mu\text{-}\lim_{n \in \mathbb{N}} f^n(x)$ is a continuous mapping of $$(\mathbb{N}^*, \sigma)$$ onto $$(\omega f(x), f)$$. 

\( (X, f) \) is called weakly incompressible if for every closed \( K \subseteq X \) with \( \emptyset \neq K \neq X \), we have \( f(K) \not\subseteq \text{Int}(K) \).
\((X, f)\) is called \textit{weakly incompressible} if for every closed \(K \subseteq X\) with \(\emptyset \neq K \neq X\), we have \(f(K) \nsubseteq \text{Int}(K)\).

\textbf{Theorem}

- (Bowen, 1975) A metrizable dynamical system is an abstract omega-limit set if and only if it is weakly incompressible.
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**Theorem**

- (Bowen, 1975) A metrizable dynamical system is an abstract omega-limit set if and only if it is weakly incompressible.
- (B., 2016) A dynamical system of weight \( \leq \aleph_1 \) is an abstract omega-limit set if and only if it is weakly incompressible.
\((X, f)\) is called *weakly incompressible* if for every closed \(K \subseteq X\) with \(\emptyset \neq K \neq X\), we have \(f(K) \not\subseteq \text{Int}(K)\).

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- (Bowen, 1975) A metrizable dynamical system is an abstract omega-limit set if and only if it is weakly incompressible.
- (B., 2016) A dynamical system of weight ≤ ℵ₁ is an abstract omega-limit set if and only if it is weakly incompressible.
- (B., 2015) The same cannot necessarily be said for spaces of weight ℵ₂, even when the Continuum Hypothesis fails badly.
- (B., 2016) Every dynamical system of weight ≤ ℵ₁ is a continuous image of a subsystem of (N*, σ).
And it’s useful too!

These universal properties of $\beta N$ have many applications:

Theorem (Auslander, 1960)
In every dynamical system, every point is proximal to a minimal point.

Theorem (B. (2015), Oprocha (2015))
If $\left( X, f \right)$ is a metrizable dynamical system (with metric $d$), then the following are equivalent:

1. For any sequence $\xi$ of points in $X$, any ultrafilter $\mu$, and any $\varepsilon > 0$, there is some $x \in X$ such that $\mu$-almost-always, $d(\xi(n), f^n(x)) < \varepsilon$.

2. $X$ has a dense set of minimal points.
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Taking ultrafilter limits of ultrafilters

For $\mu, \nu \in \beta \mathbb{N}$, define

$$\mu + \nu = \mu \lim_{n \in \mathbb{N}} \sigma^n(\nu).$$
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- From a measure-theoretic point of view, $\mu + \nu$ is simply the convolution $\mu \ast \nu$ of the finitely additive measures $\mu$ and $\nu$. 

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- From a measure-theoretic point of view, $\mu + \nu$ is simply the convolution $\mu * \nu$ of the finitely additive measures $\mu$ and $\nu$.
- If $m, n \in \mathbb{N}$, then $\mu_m + \mu_n = \mu_{m+n}$. Thus we may think of this operation as generalizing the usual addition operation on $\mathbb{N}$.
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- If $m, n \in \mathbb{N}$, then $\mu_m + \mu_n = \mu_{m+n}$. Thus we may think of this operation as generalizing the usual addition operation on $\mathbb{N}$.
- This operation is associative, so $(\beta N, +)$ is a semigroup, and $(\mathbb{N}^*, +)$ is a subsemigroup of it.
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- If $m, n \in \mathbb{N}$, then $\mu_m + \mu_n = \mu_{m+n}$. Thus we may think of this operation as generalizing the usual addition operation on $\mathbb{N}$.
- This operation is associative, so $(\beta\mathbb{N}, +)$ is a semigroup, and $(\mathbb{N}^*, +)$ is a subsemigroup of it.
- An alternative (but equivalent) definition:

$$ (\mu + \nu)(A) = 1 \iff \mu(\{n : \nu(A - n) = 1\}) = 1. $$
As in any semigroup, $\mu \in \beta N$ is called *idempotent* if $\mu + \mu = \mu$. 
As in any semigroup, \( \mu \in \beta N \) is called *idempotent* if \( \mu + \mu = \mu \). In this context, \( \mu \) is idempotent if and only if for any \( \mu \)-large set \( A \), the set of all \( n \) such that \( A - n \) is \( \mu \)-large is \( \mu \)-large.

Theorem (Numakura (1952) and Ellis (1958)) \( \beta N \) contains idempotents. More precisely, if \( K \) is any closed subset of \( \beta N \) that is closed under \( \sigma \), then \( K \) contains an idempotent ultrafilter (in fact, it contains \( 2^{2^{\aleph_0}} \) of them).
As in any semigroup, $\mu \in \beta \mathbb{N}$ is called *idempotent* if $\mu + \mu = \mu$. In this context, $\mu$ is idempotent if and only if for any $\mu$-large set $A$, the set of all $n$ such that $A - n$ is $\mu$-large is $\mu$-large.

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An application: Hindman’s theorem

If $\langle a_n \rangle_{n \in \mathbb{N}}$ is a sequence of natural numbers, define

$$FS(\langle a_n \rangle_{n \in \mathbb{N}}) = \{a_{n_0} + a_{n_1} + \cdots + a_{n_k} : n_0 < n_1 < \cdots < n_k\}.$$
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Theorem (Hindman, 1974)

Suppose the sets \( A_0, A_1, \ldots, A_n \) form a partition of \( \mathbb{N} \). There is some \( i \leq n \), and some infinite sequence \( \langle a_n \rangle_{n \in \mathbb{N}} \) of members of \( A_i \), such that \( FS(\langle a_n \rangle_{n \in \mathbb{N}}) \subseteq A_i \).
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The idea of the proof is simple: Let \( \mu \) be an idempotent ultrafilter, and pick \( i \leq n \) so that \( \mu(A_i) = 1 \). The sequence \( \langle a_n \rangle_{n \in \mathbb{N}} \) can be found using a (surprisingly short) argument reminiscent of the Poincaré recurrence theorem.
More Ramsey theory

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Idempotent ultrafilters, and especially the *minimal idempotents*, play a critical part in almost all of these proofs.
Minimal idempotents

There is a natural ordering of the idempotents of $\beta\mathbb{N}$: If $\mu$ and $\nu$ are both idempotents, then

$$\mu \leq \nu \iff \mu = \mu + \nu \iff \mu \in \omega_\sigma(\nu) \iff \omega_\sigma(\mu) \subseteq \omega_\sigma(\nu).$$
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**Question (Hindman-Strauss, 1998)**

*Does* $(\beta N, +)$ *contain an idempotent that is both minimal and maximal with respect to this order?*
An affirmative answer

Theorem (Zelenyuk (2014) and B. (2015))

There is an idempotent ultrafilter that is both minimal and maximal.

*Proof sketch:* The idea is to find a minimal idempotent $\mu$ such that $\omega_\sigma(\mu)$ is sufficiently “far away” from all other idempotents in $\mathbb{N}^*$.
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\[
\begin{array}{c}
\mathbb{N}^* \\
\downarrow
\end{array}
\begin{array}{c}
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\end{array}
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- In other words, we find an idempotent $\mu$ such that $\omega_\sigma(\mu)$ is disjoint from $\omega_\sigma(\nu)$ for every $\nu$ not already in $\omega_\sigma(\mu)$.
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![Diagram](image-url)
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- In fact, we arrange things so that $\omega_\sigma(\mu)$ is disjoint from the closure of any countable set in $\mathbb{N}^* - \omega_\sigma(\mu)$.
- This is done by building on a theorem of Kunen from 1980, where he showed that there are points in $\mathbb{N}^*$ with this property.
Thank you for listening