PRESERVATION AND DESTRUCTION IN SIMPLE REFINEMENTS

WILLIAM R. BRIAN

Abstract. If \( \sigma \) is a topology on \( X \) and \( A \subseteq X \), we let \( \langle \sigma, A \rangle \) denote the topology generated by \( \sigma \) and \( A \), i.e., the topology with \( \sigma \cup \{A\} \) as a subbasis. Any refinement of a topology obtained like this – by declaring just one new set to be open – we call simple. The present paper investigates the preservation of various properties in simple refinements. The locally closed sets (sets open in their closure) play a crucial role here: it turns out that many properties are preserved in a simple refinement by \( A \) if and only if \( A \) is locally closed. We prove this for the properties of regularity, completely regularity, (complete) metrizability, and (complete) ultrametrizability. We also show that local compactness is preserved in a simple refinement by \( A \) if and only if both \( A \) and its complement are locally closed.

1. Introduction

Possibly the most widely known result about changing a topology is the following: given a compact Hausdorff topology on a set \( X \), any finer topology on \( X \) is non-compact, and any coarser topology is non-Hausdorff. This can be rephrased by saying that compact Hausdorff spaces are “minimally Hausdorff” and “maximally compact”. Many other results are also known about spaces that have a certain property maximally or minimally. This has been a lively area of study, and a thorough summary of results like this can be found at the end of [14].

We change the pattern of these results as follows. Instead of looking for particular spaces in which a property \( P \) cannot be preserved under refinement, we look at arbitrary spaces satisfying \( P \). In some refinements \( P \) might be preserved and in others \( P \) might be destroyed, and our basic question is: which ones are which?

Mostly, we restrict ourselves to a special kind of topological refinement. If \( \sigma \) is a topology on a set \( X \) and \( A \subseteq X \), we let \( \langle \sigma, A \rangle \) denote

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the topology generated by \( \sigma \) and \( A \), i.e., the topology with \( \sigma \cup \{A\} \) as a subbasis. Any refinement of a topology obtained like this – by declaring just one new set to be open – we call simple. Given a property \( P \), we will try to determine for which sets \( A \) a simple refinement by \( A \) preserves or destroys the property \( P \).

The locally closed sets play an important role in these results. A set \( A \subseteq X \) is called locally closed (with respect to some topology on \( X \)) if it satisfies any of the following equivalent properties:

**Lemma 1.1.** Let \( X \) be a topological space and \( A \subseteq X \). The following are equivalent:

1. \( A \) is open in its closure.
2. \( A \) is the intersection of an open set and a closed set.
3. \( A = U \setminus V \) with \( U \) and \( V \) either both open or both closed.
4. If \( x \in A \), there is some open \( U \subseteq X \) with \( x \in U \) such that \( U \cap A \) is closed in \( U \).

In what follows, we will see that many nice properties of a topology \( \sigma \) are preserved in \( \langle \sigma, A \rangle \) if \( A \) is locally closed, and are destroyed if \( A \) is not locally closed.

We will often need to consider several topologies on a single set. To avoid confusion, we will write \( \overline{A}^\sigma \) to mean the closure of \( A \) with respect to the topology \( \sigma \), and we will use other similar conventions for other topological operations. If \( \sigma \) is a topology on \( X \) and \( \mathcal{A} \) is a collection of subsets of \( X \), then \( \langle \sigma, \mathcal{A} \rangle \) is the topology with \( \sigma \cup \mathcal{A} \) as a subbasis (and \( \langle \sigma, \mathcal{A} \rangle \) is an abbreviation for \( \langle \sigma, \{A\} \rangle \) when \( A \subseteq X \)). If \( \sigma \) and \( \tau \) are two topologies on \( X \) and \( \sigma \subseteq \tau \), then \([\sigma, \tau]\) denotes the set of all topologies \( \alpha \) on \( X \) such that \( \sigma \subseteq \alpha \subseteq \tau \). This notation arises from the fact that we consider \([\sigma, \tau]\) to be an interval in the lattice of all topologies on \( X \). If \( B \subseteq X \), we let \( \sigma \upharpoonright B = \{U \cap B : U \in \sigma\} \) denote the subspace topology that \( B \) inherits from \( \sigma \).

The following few lemmas are not difficult to prove, and they will help us keep track of what is going on as we move between \( \sigma \) and \( \langle \sigma, A \rangle \):

**Lemma 1.2.** Let \( \sigma \) and \( \tau \) be topologies on a set \( X \) with \( \sigma \subseteq \tau \) and let \( B \subseteq X \). Then \( \overline{B}^\sigma \subseteq \overline{B}^\tau \).

**Lemma 1.3.** Let \( \sigma \) be a topology on a set \( X \) and let \( A \subseteq X \). If \( x \notin A \), then \( \{U \in \sigma : x \in U\} \) is a neighborhood basis for \( x \) in \( \langle \sigma, A \rangle \). If \( x \in A \) then \( \{U \cap A : x \in U \in \sigma\} \) is a neighborhood basis for \( x \) in \( \langle \sigma, A \rangle \). If \( \mathcal{B} \) is a basis for \( \sigma \), then “\( \sigma \)” can be replaced with “\( \mathcal{B} \)” in the definitions of these neighborhood bases.

**Proof.** This follows directly from the fact that \( \sigma \cup \{A\} \) is a subbasis for \( \langle \sigma, A \rangle \). \( \square \)
Lemma 1.4. Let $\sigma$ be a topology on a set $X$ and let $A \subseteq X$. If $B \subseteq X$ then $\{U \cap B : U \in \sigma \cup \{A\}\}$ is a subbasis for $\langle \sigma, A \rangle \upharpoonright B$. Moreover, $\langle \sigma, A \rangle \upharpoonright B = \langle \sigma \upharpoonright B, A \cap B \rangle$. In other words, the operations of taking subspaces and taking simple refinements commute.

Proof. Because $\sigma \cup \{A\}$ is a subbasis for $\langle \sigma, A \rangle$, $\{U \cap B : U \in \sigma \cup \{A\}\}$ is a subbasis for $\langle \sigma, A \rangle \upharpoonright B$. To prove the second claim, note that $\{U \cap B : U \in \sigma \cup \{A\}\} = \{U \cap B : U \in \sigma \cup \{A \cap B\}\}$, and the latter is by definition a subbasis for $\langle \sigma \upharpoonright B, A \cap B \rangle$. □

Lemma 1.5. Let $\sigma$ be a topology on a set $X$ and let $A \subseteq X$. If $B \subseteq A$ or $B \subseteq X \setminus A$, then $\sigma \upharpoonright B = \langle \sigma, A \rangle \upharpoonright B$.

Proof. This is a special case of Lemma 1.4. □

Lemma 1.6. Let $\sigma$ be a topology on a set $X$ and let $A \subseteq X$. If $U \subseteq A$ then $\overline{U}^{\sigma} = \overline{U}^{\langle \sigma, A \rangle}$.

Proof. That $\overline{U}^{\sigma} \supseteq \overline{U}^{\langle \sigma, A \rangle}$ follows from Lemma 1.2. That $\overline{U}^{\sigma} \subseteq \overline{U}^{\langle \sigma, A \rangle}$ follows from Lemma 1.3 and the definition of a closure. □

Before beginning, we note that many important properties are trivially preserved in any refinement: the $T_0$ - $T_{2\frac{1}{2}}$ axioms, the completely Hausdorff property, (total) disconnectedness, and scatteredness, to name a few. Other important properties are preserved in any simple refinement: weight, $\pi$-weight, character, and hereditary separability, for example. Some properties are destroyed in every refinement: namely, any property beginning with the word “maximally”. These trivial examples having been mentioned, we leave them here in the introduction and go on to look at properties that are preserved in some simple refinements and destroyed in others.

2. Separation properties

In this section we will prove that regularity and complete regularity are preserved in a simple refinement by $A$ if and only if $A$ is locally closed. Here we do not use assume that a regular (completely regular, normal, hereditarily normal) space is Hausdorff; we follow the convention that a space is $T_3$ ($T_{3\frac{1}{2}}$, $T_4$, $T_5$) when it is both regular (completely regular, normal, hereditarily normal) and Hausdorff.

Theorem 2.1. If $\sigma$ is a topology on a set $X$ and $A$ any subset of $X$ that is not locally closed, then $\langle \sigma, A \rangle$ is not regular. If $\sigma$ is regular then the converse also holds: i.e., $\langle \sigma, A \rangle$ is regular if and only if $A$ is locally closed with respect to $\sigma$. 
Proof. Let $\sigma$ be a topology on $X$ and let $A \subseteq X$ be any set that is not locally closed. By Lemma 1.1, there is some $x \in A$ such that for any $U \in \sigma$, $U \cap A$ is not closed in $U$ with respect to $\sigma$. In $\langle \sigma, A \rangle$, $A$ is an open set containing $x$. We will show that $\langle \sigma, A \rangle$ is not regular by showing that for any $U \in \langle \sigma, A \rangle$ with $x \in U$, we have $\overline{U}^{(\sigma,A)} \not\subseteq A$.

Let $U \in \langle \sigma, A \rangle$ with $x \in U \subseteq A$. By Lemma 1.6, $\overline{U}^{(\sigma,A)} = \overline{U}^\sigma$. By Lemma 1.5, we must have $U_0 = U_0 \cap A$ for some $U_0 \in \sigma$. By our choice of $x$, $U$ is not closed in $\sigma | U_0$. Since $U$ is not a closed subset of $U_0$, we have $U_0 \cap \overline{U}^{(\sigma,A)} = U_0 \cap \overline{U}^\sigma \neq U$. Since $U = U_0 \cap A$, this means that $U_0 \cap \overline{U}^{(\sigma,A)}$ contains a point of $U_0 \setminus A$. This shows that $\langle \sigma, A \rangle$ is not regular.

For the second claim, suppose $\sigma$ is regular and let $A$ be any subset of $X$ that is open in its closure with respect to $\sigma$. We must show that $\langle \sigma, A \rangle$ is regular.

By Lemma 1.3, $\mathcal{B} = \sigma \cup \{V \cap A : V \in \sigma\}$ is a basis for $\langle \sigma, A \rangle$. Let $U \subseteq X$ and $x \in X$ with $x \in U \in \mathcal{B}$. It suffices to show that there is a $W \in \mathcal{B}$ such that $x \in W \subseteq \overline{W}^{(\sigma,A)} \subseteq U$.

If $U \in \sigma$ then (because $\sigma$ is regular) there is some $W \in \sigma$ such that $x \in W \subseteq \overline{W}^\sigma \subseteq U$.

In the remaining case, there is some $V \in \sigma$ such that $U = U \cap A$. In particular, we have $x \in A$. By assumption, there is some $W_0 \in \sigma$ such that $A \cap W_0$ is a closed subset of $W_0$ with respect to the topology $\sigma | W_0$. By the regularity of $\sigma$, there is some $W \in \sigma$ such that $x \in W \subseteq \overline{W}^\sigma \subseteq W_0$. Then $x \in W \cap A = \langle \sigma, A \rangle$, so we are done if we can prove that $\overline{W \cap A}^{(\sigma,A)} \subseteq U$.

To compute $\overline{W \cap A}^{(\sigma,A)}$, note that $\overline{W \cap A}^{(\sigma,A)} \subseteq \overline{W}^\sigma \subseteq W_0$, so that the closure of $W \cap A$ with respect to $\langle \sigma, A \rangle$ is the same whether it is computed in $X$ or in the subspace $W_0$ of $X$. Working in $W_0$: since $W_0 \cap A$ is closed in $W_0$ with respect to $\sigma$, it follows from Lemma 1.6 that $\overline{W \cap A}^{(\sigma,A)} = \overline{W \cap A'} = A \cap \overline{W}^\sigma \subseteq A \cap W_0 \subseteq U$. \hfill \Box

**Corollary 2.2.** Let $\sigma$ be a regular topology on $X$ and let $A$ be any collection of locally closed subsets of $X$. Then $\langle \sigma, A \rangle$ is regular.

**Proof.** Note that the intersection of any two locally closed sets is locally closed. Therefore we may assume without loss of generality that $A$ is closed under finite intersections and that $X \in A$. Let $x \in X$ and let $U \in \langle \sigma, A \rangle$ with $x \in U$. By our assumptions about $A$, we can write $U = V \cap A$ for some $V \in \sigma$ and $A \in A$. If $\sigma' = \langle \sigma, A \rangle$, then by Theorem 2.1 there is some $W \in \sigma'$ with $x \in W \subseteq \overline{W}^{\sigma'} \subseteq U$. Since
Corollary 2.3. Let $\sigma, \tau$ be topologies on a set $X$. Then every topology in the interval $[\sigma, \tau]$ is regular if and only if $\sigma$ is regular and every $U \in \tau$ is locally closed with respect to $\sigma$.

Proof. If some $A \in \tau$ is not locally closed then $\langle \sigma, A \rangle \in [\sigma, \tau]$ is not regular by Theorem 2.1. If every $A \in \tau$ is locally closed and if $\alpha \in [\sigma, \tau]$, then $\alpha \setminus \sigma \subseteq \tau$ and $\alpha = \langle \sigma, \alpha \setminus \sigma \rangle$ is regular by Corollary 2.2. □

Recall that a space $X$ is called maximal if $X$ is dense-in-itself and no refinement of $X$ is dense-in-itself. $X$ is submaximal if every subset of $X$ is locally closed. It has been shown (see [11] or [13]) that $X$ is maximal if and only if it is dense-in-itself, submaximal, and extremally disconnected. These spaces have been thoroughly investigated, for example in [10], [3], and [2]. The following corollary to Theorem 2.1 gives an equivalent formulation of submaximality for regular spaces.

Corollary 2.4. Let $\sigma$ be a regular topology on a set $X$. Then $\sigma$ is submaximal if and only if every refinement of $\sigma$ is regular.

Proof. Use Corollary 2.3 with $\tau$ the discrete topology. □

One implication of Corollary 2.4 is that submaximal regular spaces remain submaximal under refinement. In fact, this is true of all submaximal spaces by a result of Arhangel’skii and Collins (see [2], Corollary 2.9; see also the stronger Theorem 2.13).

There is a class of submaximal spaces that is easy to describe: every scattered space of rank 1 or 2 is submaximal (and these are the only submaximal scattered spaces; see Corollary 1.7 in [2]). However, it should be noted that non-scattered examples have been constructed: see [10]; see [5] and [6], or Example 3.10 in [2]; or see Section 2 of [1]. All non-scattered examples of submaximal spaces are constructed with some version of the Axiom of Choice, typically transfinite recursion. This is necessary because the topology of any such must contain the base of a free ultrafilter (see [7]).

It turns out that Theorem 2.1 and Corollaries 2.2, 2.3, and 2.4 all have exact analogues when regularity is replaced by complete regularity.

Theorem 2.5. Let $\sigma$ be a completely regular topology on a set $X$ and let $A \subseteq X$. Then $\langle \sigma, A \rangle$ is completely regular if and only if $A$ is locally closed with respect to $\sigma$. 

Proof. The “only if” direction follows from Theorem 2.1. For the “if”
direction, let $U$ be basic open neighborhood of $x$ in $\langle \sigma, A \rangle$ (that is, “basic” as determined by the subbasis $\sigma \cup \{A\}$).

First suppose that there is some $V \subseteq U$ such that $x \in V \in \sigma$. Because $\sigma$ is completely regular, there is a function $f : X \to [0, 1]$ that is 1 on $\{x\}$ and 0 on $X \setminus V$, and $f$ is continuous with respect to $\sigma$. Automatically, $f$ is continuous with respect to the finer topology $\langle \sigma, A \rangle$ as well. Thus, in $\langle \sigma, A \rangle$, $x$ is functionally separated from $X \setminus U$.

Next suppose that $U = A \cap V$ for some $V \in \sigma$. Since $A$ is locally closed with respect to $\sigma$, there is some $V' \in \sigma$ such that $A$ is a relatively closed subset of $V'$; i.e., $A \cap V' = \overline{A} \cap V'$. Replacing $V$ with $V \cap V'$ if necessary, we may assume that $U = A \cap V = \overline{A} \cap V$. As $V \in \sigma$ and $\sigma$ is regular, there is some $W \in \sigma$ such that $x \in W \subseteq \overline{W} \subseteq V$.

As $\sigma$ is completely regular, there is a $\sigma$-continuous function $f : X \to [0, 1]$ that is 1 on $\{x\}$ and 0 on $X \setminus W$. Let $g : X \to [0, 1]$ be the function that is equal to $f$ on $W \cap A$ and equal to 0 elsewhere. Since $W \subseteq V$, $W \cap A \subseteq U$. Therefore $g$ is equal to 1 on $\{x\}$ and to 0 on $X \setminus U$. In other words, $g$ is a functional separation of $x$ from $X \setminus U$, provided that $g$ is continuous with respect to $\langle \sigma, A \rangle$. It remains to check that this is true.

By definition, $g$ is identically 0, hence continuous, on $X \setminus W$. To prove $g$ continuous on $X$, it therefore suffices to check that $g$ is continuous (with respect to $\langle \sigma, A \rangle$) on $Y = \overline{W}$. Continuity then follows from the well-known “pasting lemma”, since $X \setminus W$ and $Y$ are both closed in $\langle \sigma, A \rangle$ and $Y \cup (X \setminus W) = X$.

By our choice of $W$ and $V$, $Y \subseteq V$ and $A \cap V = \overline{A} \cap V$. Thus

$$A \cap Y = A \cap V \cap Y = \overline{A} \cap V \cap Y = \overline{A} \cap Y$$

is closed in $Y$ with respect to $\sigma$. Hence $Y \cap A$ is clopen in $Y$ with respect to $\langle \sigma, A \rangle$. Therefore, to check that $g$ is $\langle \sigma, A \rangle$-continuous on $Y$, it suffices to check that $g$ is continuous on $Y \cap A$ and on $Y \setminus A$. This is trivial: $g = f$ on $Y \cap A$, and $g$ is identically 0 on $Y \setminus A$. □

Corollary 2.6. Let $\sigma$ be a completely regular topology on $X$ and let $A$ be any collection of locally closed subsets of $X$. Then $\langle \sigma, A \rangle$ is completely regular.

Proof. The argument is essentially identical to that in Corollary 2.2. The only differences are that we replace closed neighborhoods with separating functions and Theorem 2.1 with Theorem 2.5. □

Corollary 2.7. Let $\sigma, \tau$ be topologies on a set $X$. Then every topology in the interval $[\sigma, \tau]$ is completely regular if and only if $\sigma$ is completely regular and every $U \in \tau$ is locally closed with respect to $\sigma$. □
Corollary 2.8. Let $\sigma$ be a completely regular topology on a set $X$. Then $\sigma$ is submaximal if and only if every refinement of $\sigma$ is completely regular.

Corollaries 2.4 and 2.8 taken together raise the following question:

Question 2.9. Is there a submaximal space that is regular but not completely regular?

The following partial result is all that is known at present:

Proposition 2.10. If $X$ is a submaximal scattered space, then $X$ is completely regular.

Proof. As mentioned above, a scattered space is submaximal if and only if it has rank 1 or 2. Rank 1 spaces are discrete and so $T_{3\frac{1}{2}}$. If $X$ has rank 2, it is easy to check that $X$ has small inductive dimension 0 (i.e., it has a basis of clopen sets). This makes $X$ completely regular. □

In contrast to Proposition 2.10, we show in the following example that a submaximal scattered space need not be normal.

Example 2.11. Let $X$ denote the “one-point Lindelöfication” of the discrete space of size $\aleph_1$: that is, $X$ is the space $\omega_1 \cup \{\ast\}$, where every point of $\omega_1$ is isolated and basic open neighborhoods of $\ast$ are of the form $(\omega_1 \setminus \alpha) \cup \{\ast\}$, $\alpha < \omega_1$. Let $Y$ denote the convergent sequence $\omega + 1$. Let $Z = X \times Y \setminus \{(\ast, \omega)\}$. $Z$ is a scattered space of rank 2, and is therefore submaximal. Furthermore, $Z$ is $T_{3\frac{1}{2}}$ by Proposition 2.10. However, $Z$ is not normal. The proof of this is left as an exercise for the reader (hint: it is essentially the same as the proof that the deleted Tychonov plank is not normal).

In light of Theorems 2.1 and 2.5, one might wonder whether normality is also preserved by declaring some locally closed set to be open. As the following example shows, this is not the case, and even replacing “locally closed” by “closed” or “finite” does not suffice to preserve normality.

Example 2.12. Let $T = (\omega_1 + 1) \times (\omega + 1)$ with the usual product topology $\sigma$ (i.e., $T$ is the Tychonoff plank), and let $x$ be the point $(\omega_1, \omega)$. If $\tau = \langle \sigma, \{x\}\rangle$, then the set $T' = T \setminus \{x\}$ is clopen with respect to $\tau$, and its topology is just the subspace topology inherited from $\sigma$. $T'$ is a well-known example of a space that is $T_{3\frac{1}{2}}$ but not $T_4$.

While normality is not preserved by declaring closed sets open, hereditary normality is:
Proposition 2.13. Suppose $\sigma$ is a hereditarily normal topology on $X$ and $A \subseteq X$ is closed with respect to $\sigma$. Then $\langle \sigma, A \rangle$ is hereditarily normal.

Proof. Because $A$ is closed in $\sigma$, $A$ is clopen in $\tau = \langle \sigma, A \rangle$. By Lemma 1.5, $A$ and $X \setminus A$ are both hereditarily normal with respect to $\tau$. □

The previous example and proposition are really one instance of a broader set of results. The general case is: (1) if $P$ is a property that is hereditary for clopen sets but not open sets, then $P$ is not preserved in simple refinements by closed sets (2) if $P$ is any property hereditary for open and closed sets and preserved under finite sums, then $P$ is preserved by declaring a closed set to be open. For another instance of results of this kind, replace “normal” and “hereditarily normal” with “Lindelöf” and “hereditarily Lindelöf”, respectively.

Question 2.14. If $\sigma$ is a hereditarily normal (or $T_5$) topology on $X$ and $A \subseteq X$ is locally closed, is $\langle \sigma, A \rangle$ hereditarily normal ($T_5$)? If not, then what conditions on $A$ are necessary and sufficient to preserve hereditary normality (the $T_5$ property)?

3. Metric properties

In this section we consider the properties of metrizability and complete metrizability. Once again, we find that if these properties hold in $\sigma$ then they hold in $\langle \sigma, A \rangle$ if and only if $A$ is locally closed.

Theorem 3.1. Let $\sigma$ be a metrizable topology on a set $X$.

(1) If $A \subseteq X$, then $\langle \sigma, A \rangle$ is metrizable if and only if $A$ is locally closed with respect to $\sigma$.

(2) Let $\mathcal{A}$ be a countable collection of subsets of $X$, each of which is locally closed with respect to $\sigma$. Then $\langle \sigma, \mathcal{A} \rangle$ is metrizable.

Proof. If $A \subseteq X$ is not locally closed, then $\langle \sigma, A \rangle$ is not $T_3$ by Theorem 2.1. To prove (1), then, it suffices to show that if $A$ is locally closed then $\langle \sigma, A \rangle$ is metrizable. This is a special case of (2), so we will prove (2).

Let $\tau = \langle \sigma, \mathcal{A} \rangle$. By the Nagata-Smirnov Metrization Theorem, $\tau$ is metrizable if and only if it is $T_3$ and has a countably locally finite base (a base that can be written as a countable union of locally finite collections of sets). By Corollary 2.2, $\tau$ is $T_3$. Therefore we need to show that $\tau$ has a countably locally finite base.

Suppose $\bigcup_{n \in \omega} \mathcal{B}_n$ is a base for $\sigma$, with each $\mathcal{B}_n$ locally finite. Let $\mathcal{A}'$ be the set of all finite intersections of sets in $\mathcal{A}$. Note that $\mathcal{A}'$ is still countable, and write $\mathcal{A}' = \{ A_n : n \in \omega \}$. For each $m, n \in \omega$, let
\[ C_{m,n} = B_m \cup \{ U \cap A_n : U \in B_m \}. \] Clearly \( \bigcup_{m \in \omega} \bigcup_{n \in \omega} C_{m,n} \) is a basis for \( \tau \). If \( U \) is a neighborhood of \( x \) in \( \sigma \) meeting only finitely many elements of \( B_m \), then \( U \) is a neighborhood of \( x \) in \( \tau \) meeting only finitely many elements of \( C_{m,n} \) (it meets at most twice as many). Therefore \( \tau \) has a countably locally finite base. \[ \square \]

Remark 3.2. Theorem 3.1 has a more direct and “hands-on” proof that avoids the power of the Nagata-Smirnov Metrization Theorem. It is a longer and messier process to check all the details of this proof, but we give a rough outline here for the interested reader.

Let \( X \) be a metrizable space with topology \( \sigma \), let \( A \subseteq X \), and let \( d \) be a bounded metric on \( X \) that does not take any values larger than 1. Define

\[
\delta_A(x,y) = \min \left\{ 1, \inf \left\{ d(x,z) + d(z,y) : z \in \overline{A} \setminus A \right\} \right\},
\]

\[
\delta(x,y) = \begin{cases} 
  d(x,y) & \text{if } x, y \in A \text{ or } x, y \notin A \\
  \delta_A(x,y) & \text{otherwise.}
\end{cases}
\]

One can prove that \( \delta \) is a metric on \( X \). Setting

\[ B = \text{Int}_{\sigma}(A) \cup \text{Int}_{\sigma|\overline{A}}(A), \]

it is possible to show that \( \delta \) generates the topology \( \langle \sigma, B \rangle \). Finally, one can prove that \( A = B \) if and only if \( A \) is locally closed.

Corollary 3.3. Let \( \sigma \) be a topology on \( X \). The following are equivalent:

1. \( \sigma \) is metrizable and scattered with rank at most 2.
2. \( \sigma \) is metrizable and submaximal.
3. Any simple refinement of \( \sigma \) is metrizable.
4. For any countable \( A \subseteq \mathcal{P}(X) \), \( \langle \sigma, A \rangle \) is metrizable.

Proof. By Corollary 1.7 in [2], a scattered space is submaximal if and only if it has rank at most 2; thus (1) implies (2). That (2) implies (4) is immediate from Theorem 3.1(2) and the definition of submaximality. Clearly (4) implies (3).

To see that (3) implies (1), first note that (3) automatically gives that \( \sigma \) is metrizable by taking \( A = X \). If (3) holds and \( \sigma \) is scattered, then \( \sigma \) must have rank at most 2, since otherwise \( \sigma \) is not submaximal and (by Theorem 3.1) has a non-metrizable simple refinement. To complete the proof, then, it suffices to show that (3) implies \( \sigma \) is scattered.

Suppose \( \sigma \) is not scattered and let \( P \) be the dense-in-itself part of \( X \). If \( x \in P \), we can use metrizability to find a sequence \( \langle (x_n, U_n) : n \in \omega \rangle \) such that \( x_n \to x \), the \( U_n \) have pairwise disjoint closures, and for each \( n \), \( U_n \) is an open neighborhood of \( x_n \) with \( U_n \subseteq P \). Then, for each \( n \), we may find a sequence \( \langle x_m^n : m \in \omega \rangle \) of points in \( U_n \) converging to
Let $A = \{x_m^n : m, n \in \omega \} \cup \{x \}$. By our choice of the $U_n$, $\overline{A} = A \cup \{x_n : n \in \omega \}$. Every neighborhood of $x$ contains some (cofinitely many) $x_n$, so $A$ is not open in its closure. By Theorem 3.1(1), $\langle \sigma, A \rangle$ is not metrizable. \qed

The next result shows that Corollaries 2.4 and 2.8 have no analogue for metrizability (or even first countability). Since there are metrizable submaximal spaces, it also shows that the countability condition cannot be removed from Theorem 3.1(2).

**Proposition 3.4.** If $X$ is a non-discrete first countable $T_1$ space, there is a collection $A$ of subsets of $X$ with $|A| = \aleph_1$ such that $\langle \sigma, A \rangle$ is not first countable.

**Proof.** Let $x$ be a non-isolated point with respect to $\sigma$. If $Y = X \setminus \{x\}$, then $\mathcal{F} = \{U \cap Y : x \in U \in \sigma\}$ is a filter on $Y$ (because $x$ is non-isolated) and is non-principal (because $\sigma$ is $T_1$). Using transfinite recursion, it is easy to find a set $A_0$ of subsets of $Y$ with $|A_0| = \aleph_1$ such that $\mathcal{F} \cup A_0$ is a filter base on $Y$, and the filter generated by $\mathcal{F} \cup A_0$ is not countably based. It follows that $x$ does not have a countable neighborhood basis in the topology

$$
\tau = \langle \sigma, \{A \cup \{x\} : A \in A_0\} \rangle,
$$

so $\tau$ is not first countable. Setting $A = \{A \cup \{x\} : A \in A_0\}$ completes the proof. \qed

In the previous proof, if we did not care about the cardinality of $A$ then we could have extended $\mathcal{F}$ to an ultrafilter and declared every point other than $x$ to be isolated. Topologies of this kind, called “ultratopologies”, have been studied before (see, e.g., [14]). More formally, an ultratopology is a topology with exactly one non-isolated point, where the neighborhoods of that point induce an ultrafilter on the remaining points. Fröhlich was the first to show that every (non-discrete) topology can be refined to an ultratopology (see [9]). Given this result, the following version of Proposition 3.4 holds: if $P$ is a property not possessed by any ultratopology and if $X$ is a non-discrete space with property $P$, there is a refinement of $X$ without property $P$.

In the next theorem we look at complete metrizability, and for this we make use of the strong Choquet game on $X$ (see [4] or Chapter 8 of [12] for a more thorough introduction). In this game, players I and II take turns and each plays $\omega$ times:

\[
\begin{align*}
\text{I :} & \quad (x_0, U_0) \quad (x_1, U_1) \quad (x_2, U_2) \quad \ldots \\
\text{II :} & \quad V_0 \quad V_1 \quad V_2 \quad \ldots
\end{align*}
\]
Each move of I is a pair \((x_n, U_n)\) with \(x_n \in U_n \subseteq V_{n-1}\) (for convenience, set \(V_0 = X\)), and each move of II is an open set \(V_n\) such that \(x_n \in V_n \subseteq U_n\). Player II wins if and only if \(\bigcap_{n \in \omega} V_n = \bigcap_{n \in \omega} U_n \neq \emptyset\). The strong Choquet game on \(X\), when \(X\) topologized by \(\sigma\), is denoted \(G^*_\sigma\).

A strategy for I (or II) is a rule \(F\) that determines the plays of I (or II) uniquely, depending only on the previous plays of the game. Player I (or II) is said to have a winning strategy in \(G^*_\sigma\) if there is some strategy \(F\) such that if I (or II) plays according to the strategy \(F\), then I (or II) will be assured to win. A topology \(\sigma\) is strong Choquet if and only if II has a winning strategy for \(G^*_\sigma\). The following fundamental result is due to Choquet.

**Lemma 3.5** (Choquet, [4]). A topological space is completely metrizable if and only if it is metrizable and strong Choquet.

The next lemma gives us our first nontrivial example of a property of \(\sigma\) that can be preserved in \(\langle \sigma, A \rangle\) when \(A\) is not locally closed.

**Lemma 3.6.** Let \(\sigma\) be a strong Choquet topology on a set \(X\) and let \(A \subseteq X\). Then \(\langle \sigma, A \rangle\) is strong Choquet if and only if \(A\) is strong Choquet when it inherits the subspace topology from \(\sigma\).

**Proof.** Assume that \(\sigma\) and \(A\) are both strong Choquet with respect to \(\sigma\). Let \(F_X\) be a winning strategy for II in \(G^*_\sigma\) and let \(F_A\) be a winning strategy for II in \(G^*_\sigma\|A\).

We will now describe a winning strategy for II in \(G^*_\sigma|A\). Assume that I has just made his \(n\)th move \((x_n, U_n)\). If \(x_n \notin A\), then by Lemma 1.3 there is some \(U'_n\) such that \(x \in U'_n \in \sigma\). In this case, II plays according to the strategy \(F_X\), pretending that the \(n\)th move of I had been \((x_n, U'_n)\) instead of \((x_n, U_n)\). If \(x \in A\), then player II pretends that I really played the pair \((x_n, U_n \cap A)\), and that this was the first move of I in the game \(G^*_\sigma|A\). Player II then plays the remainder of the game \(G^*_\sigma|A\) as if it were the rest of this newly begun game of \(G^*_\sigma|A\), and he plays according to the strategy \(F_A\).

This strategy results in a win for II: either the whole game is played according to \(F_X\) (perhaps using a modified version of the moves of I), or at some point we change to a game of \(G^*_\sigma|A\) and play according to \(F_A\). Thus \(\langle \sigma, A \rangle\) is strong Choquet.

Conversely, suppose \(A\) is not strong Choquet in the subspace topology inherited from \(\sigma\). \(A\) is open in \(\langle \sigma, A \rangle\), and by Lemma 1.5 \(A\) has the same topology whether it is considered as a subspace of \(\sigma\) or of \(\langle \sigma, A \rangle\). Every open subset of a strong Choquet space is strong Choquet, so this shows that \(\langle \sigma, A \rangle\) is not strong Choquet.
It is worth noting that if $X$ is metrizable then the strong Choquet subspaces of $X$ are precisely the $G_δ$ subsets. This follows from Lemma 3.5 and a well-known theorem of Alexandrov, which states that the $G_δ$ sets are precisely the completely metrizable subsets of any completely metrizable space (see [12], Theorem 3.16).

**Theorem 3.7.** Let $σ$ be a completely metrizable topology on $X$ and let $A ⊆ X$. Then $⟨σ, A⟩$ is completely metrizable if and only if $A$ is locally closed with respect to $σ$.

*Proof.* By Theorem 3.1, the metrizability of $σ$ is preserved in $⟨σ, A⟩$ if and only if $A$ is locally closed with respect to $σ$. Since every closed subset of a metric space is $G_δ$, $A$ is $G_δ$. Combining this with Lemmas 3.5 and 3.6 proves the theorem. □

**Remark 3.8.** The metric defined in Remark 3.2 is not complete in general, even if $d$ was complete. Nonetheless, there is still a “hands-on” method for proving that complete metrizability is preserved in a simple refinement by a locally closed set.

Let $B = \overline{A} \setminus A$ and let $C = \text{Int}_{σB}(A)$. Let $C' = C × \{⋆\}$ be a second copy of $C$, and modify the space $X$ (topologized by $⟨σ, A⟩$) by glueing $C'$ onto $X \setminus A$ in such a way that the topology on $C' \cup X \setminus A$ is determined naturally by $σ$. Show that this is a complete metric space by explicitly defining a complete metric for it (the definition is similar to that in Remark 3.2). Then show that $X$ is a $G_δ$ subset of this space and apply Alexandrov’s Theorem.

The following corollary can be seen as a refinement of Lemma 13.2 in [12].

**Corollary 3.9.** Let $σ$ be a Polish topology on $X$ and let $A ⊆ X$. Then $⟨σ, A⟩$ is Polish if and only if $A$ is locally closed with respect to $σ$.

*Proof.* Recall that $σ$ is Polish if and only if it is separable and completely metrizable, which holds if and only if $σ$ is second countable and completely metrizable. By Theorem 3.7, it suffices to show that $⟨σ, A⟩$ is second countable whenever $σ$ is. This is obvious. □

Next, we prove analogues of Theorems 3.1 and 3.7 for ultrametrizability. Recall that, for a metric space $X$, $\text{Ind}(X) = 0$ (i.e., $X$ has large inductive dimension 0) if and only if there is a sequence $⟨U_n : n < ω⟩$ of open covers of $X$ such that

- $U_m$ refines $U_m$ whenever $m < n$.
- Each $U ∈ \bigcup_{n < ω} U_n$ is clopen.
- $\bigcup_{n < ω} U_n$ is a basis for $X$.
This is not the usual definition of the large inductive dimension, but it is equivalent for metric spaces (a general version of the equivalence is proved in [8], Theorem 7.3.1), and will suit our purposes.

**Lemma 3.10.** Let $X$ be a metric space. If $\text{Ind}_\sigma(X) = 0$ and $A \subseteq X$ is locally closed, then $\text{Ind}_{\langle \sigma, A \rangle}(X) = 0$.

**Proof.** Let $X$ be a metric space, $\sigma$ a topology on $X$ such that $\text{Ind}_\sigma(X) = 0$, and $A$ locally closed with respect to $\sigma$.

Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open covers of $X$ with the properties listed above (with respect to $\sigma$). For each $n$, let

$$\mathcal{U}_n^0 = \{U \in \mathcal{U}_n : A \cap U \text{ is not closed in } U\},$$

$$\mathcal{U}_n^1 = \{U \in \mathcal{U}_n : A \cap U \text{ is closed in } U\},$$

$$\mathcal{V}_n = \mathcal{U}_n^0 \cup \{U \cap A : U \in \mathcal{U}_n^1\} \cup \{U \setminus A : U \in \mathcal{U}_n^1\}.$$

It is clear that each $\mathcal{V}_n$ is a collection of pairwise disjoint sets, that these sets are clopen in $\langle \sigma, A \rangle$, and that $\mathcal{V}_n$ refines $\mathcal{V}_m$ whenever $m < n$.

It remains to show that $\mathcal{B} = \bigcup_{n \in \omega} \mathcal{V}_n$ is a basis for $X$. We already know that each element of $\mathcal{B}$ is clopen in $X$, so we need only show that for any $x \in U \in \langle \sigma, A \rangle$, there is some $V \in \mathcal{B}$ with $x \in V \subseteq U$.

Let $x \in X$ and let $x \in U \in \langle \sigma, A \rangle$. If $x \notin A$ then by Lemma 1.3 we may take $U \in \mathcal{U}_n$ for some $n$. If $U \in \mathcal{U}_n^0$ then $V = U \in \mathcal{V}_n$ and if $U \in \mathcal{U}_n^1$ then $V = U \setminus A \in \mathcal{V}_n$; either way, there is some $V \in \mathcal{V}_n$ such that $x \in V \subseteq U$. If $x \in A$ then by Lemma 1.3 we may take $U = U_0 \cap A$ with $U_0 \in \mathcal{U}_n$ for some $n$. Since $A$ is locally closed, we may choose $U_0$ small enough ($n$ large enough) so that $A \cap U_0$ is closed in $U_0$. Then $U_0 \in \mathcal{U}_n^1$ and $V = A \cap U_0 \in \mathcal{V}_n$; we have $x \in V \subseteq U$. \qed

**Theorem 3.11.** Let $\sigma$ be a (completely) ultrametrizable topology on a set $X$ and let $A \subseteq X$. Then $\langle \sigma, A \rangle$ is (completely) ultrametrizable if and only if $A$ is locally closed with respect to $\sigma$.

**Proof.** By the Morita-de Groot Theorem (for a strong version of this theorem, see Corollary 5 in [15]), a (completely) metrizable space is (completely) ultrametrizable if and only if it also has large inductive dimension 0. The result follows from this and Theorem 3.7. \qed

4. Local compactness

In this section we assume that all spaces are Hausdorff.

Local compactness is, in some sense, very close to the properties of regularity and complete regularity. It says that every point has a
neighborhood basis of compact sets (regularity and complete regularity replace “compact” with “closed” and “functionally closed”, respectively). Locally closed sets are known to play an important part in the theory of locally compact spaces:

**Lemma 4.1.** If $X$ is locally compact and $Y \subseteq X$, then $Y$ is locally compact if and only if $Y$ is locally closed.

_Proof._ See [8], Corollary 3.3.10. □

Given these observations, it is perhaps surprising that local compactness is not preserved in general by simple refinements using locally closed sets.

**Theorem 4.2.** Let $\sigma$ be a locally compact topology on a set $X$ and let $A \subseteq X$. Then $\langle \sigma, A \rangle$ is locally compact if and only if both $A$ and $X \setminus A$ are locally closed with respect to $\sigma$.

_Proof._ First, suppose that $\langle \sigma, A \rangle$ is locally compact. Since $A$ is open and $X \setminus A$ is closed with respect to $\langle \sigma, A \rangle$, both $A$ and $X \setminus A$ are locally compact (by Lemma 4.1) when given the subspace topology from $\langle \sigma, A \rangle$. By Lemma 1.5, both $A$ and $X \setminus A$ are locally compact when given the subspace topology from $\sigma$. By Lemma 4.1, both $A$ and $X \setminus A$ are locally closed with respect to $\sigma$.

For the other direction, suppose that $A$ and $X \setminus A$ are both locally closed with respect to $\sigma$.

If $x \in A$, then $A$ is a neighborhood of $x$ with respect to $\langle \sigma, A \rangle$. Moreover, the subspace topology on $A$ is the same with respect to $\sigma$ and $\langle \sigma, A \rangle$, and this topology is locally compact by Lemma 4.1. Thus $x$ is a member of a locally compact open set in $\langle \sigma, A \rangle$, and this implies that $x$ has a compact neighborhood.

If $x \notin A$ then, because $X \setminus A$ is locally closed with respect to $\sigma$, there is some $U \in \sigma$ with $x \in U$ such that $X \setminus A$ is relatively closed in $U$. $A \cap U$ is open with respect to $\sigma \upharpoonright U$. It follows from Lemma 1.4 that $\sigma \upharpoonright U = \langle \sigma, A \rangle \upharpoonright U$. Thus, in this case too, $x$ is a member of a locally compact open set. □

The next proposition can be roughly interpreted as saying that every locally compact topology is close to a compact topology in the lattice of topologies.

**Proposition 4.3.** Let $\tau$ be any locally compact topology on $X$. There is a compact topology $\sigma$ on $X$ and $A \subseteq X$ such that $\tau = \langle \sigma, A \rangle$.

_Proof._ Let $\tau$ be a non-compact, locally compact topology on $X$, let $x \in X$, and let $A \in \tau$ be a neighborhood of $x$ such that $\overline{A}$ is compact.
Let $\alpha X$ denote the one-point compactification of $X$, which is a compact topology on $X \cup \{\infty\}$. Let $\sigma$ be the quotient topology obtained from $\alpha X$ by identifying the points $x$ and $\infty$. This topology is compact, and has a natural interpretation as a topology on $X$.

We claim that $\langle \sigma, A \rangle = \tau$. If $x \neq y \in X$, it is clear that
\[ \{ U \in \tau : y \in U \not\ni x \} = \{ U \in \langle \sigma, A \rangle : y \in U \not\ni x \} . \]
Since this set is a neighborhood basis for $y$ in both $\tau$ and $\langle \sigma, A \rangle$, it remains to check that $x$ has a common neighborhood basis in $\tau$ and in $\langle \sigma, A \rangle$. By Lemma 1.3,
\[ \{ U \in \langle \sigma, A \rangle : x \in U \subseteq A \} \subseteq \{ U \in \tau : x \in U \subseteq A \} . \]
$\overline{A}$ is compact Hausdorff when it inherits its topology from $\tau$ and is still Hausdorff when it inherits its topology from $\langle \sigma, A \rangle$. Hence $\overline{A}$ inherits a coarser subspace topology from $\tau$ than from $\langle \sigma, A \rangle$. It follows that
\[ \{ U \in \langle \sigma, A \rangle : x \in U \subseteq A \} \supseteq \{ U \in \tau : x \in U \subseteq A \} . \]
This collection is a neighborhood basis for $x$ in both $\tau$ and $\langle \sigma, A \rangle$. □

In closing, we note that there are likely many more interesting topological properties that are preserved under simple refinement by a locally closed set. The small inductive dimension, for example, is reasonably well-behaved under such refinements, and the author intends to explore this topic in a forthcoming sequel to this paper. As for other properties, the list is practically infinite, and the finiteness of the author prevents a thorough investigation of them all.

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WILLIAM R. BRIAN, DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, 6823 ST. CHARLES AVE., NEW ORLEANS, LA 70118  
E-mail address: wbrian.math@gmail.com