Questions I would like to answer
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If a question on this list has been answered, I’ll make a note of it in red. I prefer this to deleting the question, because it shows that new things are being found out!

**General and set-theoretic topology**

**Question 1** (The Toronto Problem; [4], [19]). *Is it consistent to have an uncountable, non-discrete Hausdorff space that is homeomorphic to each of its uncountable subspaces?*

It is known to be consistent that there is no such space (e.g., if $2^\omega < 2^\omega_1$).

**Question 2** (The Katowice Problem; [18], [11]). *Is it consistent to have $\omega^* \approx \omega_1^*$?*

It is obviously consistent to have $\omega^* \not\approx \omega_1^*$, e.g., if $2^\omega < 2^\omega_1$.

Recall that a 2-point set is a subset of $\mathbb{R}^2$ containing exactly two points of every line.

**Question 3** (asked by Erdős; [10],[15]). *Is there a Borel 2-point set? Does ZF prove that there is a two-point set?*

ZFC proves that two-point sets exist, but it is known that the full strength of AC is not required to build them: see [16].

For the next question, $\kappa$ carries the discrete topology and $U(\kappa)$ is the set of uniform ultrafilters on $\kappa$.

**Question 4** (asked by Baker and Kunen; [1], [2]). *If $\kappa$ is a singular cardinal, is there a weak $P_{\kappa^+}$-point in $U(\kappa)$?*

In the next two questions, ordinals are assumed to carry the discrete topology.

**Question 5** ([8]). *Is it consistent to partition $\omega^\omega$ into fewer than $\mathfrak{c}$ copies of $\omega^\omega$? Is this always possible whenever $\mathfrak{c} > \text{cf}([\mathbb{R}_\omega]^{<\omega})$?*

Say that a topological space $X$ is a condensation of $Y$ if there is a continuous bijection $Y \to X$.

**Question 6** ([5], [8]). *Let $\mathcal{U}$ denote the class of perfect completely ultrametrizable spaces. Is the condensation relation well-founded on $\mathcal{U}$? What about if we restrict ourselves to spaces of size $\mathfrak{c}$?*
Set Theory and Combinatorics

Question 7 ([13], [14]). Is there a nonmeager \( P \)-filter?

A “yes” answer is easily consistent with ZFC (e.g., if there are \( P \)-points), but it is not known whether a “no” is also consistent. The latter would have large cardinal strength (at least measurables) by results in [13].

Baumgartner’s Axiom states that any two \( \aleph_1 \)-dense sets of reals are order isomorphic (where “\( \aleph_1 \)-dense” means that every interval contains \( \aleph_1 \) points). This is implied by PFA but not by MA.

Question 8 ([3], [17]). Is it possible to extend Baumgartner’s Axiom to higher cardinals? A positive answer has been announced by Itay Neeman (the extension is to \( \aleph_2 \) only, so a version of the question remains open).

Question 9. Is \( r = r_\sigma \)?

Say a set is thick if it contains arbitrarily long intervals. A thick ultrafilter means a filter consisting entirely of thick sets, such that any larger filter contains some non-thick set. It is proved in [6] that it is consistent to have thick ultrafilters be \( P \)-filters, and that the existence of a thick ultrafilter \( P \)-filter implies the existence of a \( P \)-point.

Question 10 ([6]). Is it consistent that no thick ultrafilter is a \( P \)-filter, yet \( P \)-points still exist?

Question 11 ([6]). Is it always true that some thick ultrafilter is a weak \( P \)-filter? Yes — this was answered affirmatively with the help of Jonathan Verner in [9].

A structure \( S \) is indivisible if every finite coloring of (the underlying set of) \( S \) induces a monochromatic copy of \( S \). A set or class is considered to have a digraph structure given by \( \varepsilon \).

Question 12 (from joint work with Nate Ackerman). Is \( L \) indivisible? What about \( V \)?

This question has a consistent answer of yes. In fact, \( V \) is indivisible if global choice holds, and thus \( L \) is indivisible if \( V = L \).

Let’s say that two structures are siblings if each embeds in the other.

Question 13. Up to siblinghood, classify the indivisible countable digraphs. The linear orders?

Question 14 (from Claude LaFlamme). Is it consistent that for some \( \aleph_0 < \kappa < \mathfrak{c} \), some countable digraph has exactly \( \kappa \) siblings (counting the siblings up to isomorphism)?
Question 15 (from Boaz Tsaban; see [20]). Is it consistent that \( \text{bidi} < \min\{r, d\} \)? Is it consistent that \( \max\{\text{cov}(\mathcal{M}), b\} < \text{bidi} \)?

**Topological dynamics**

**Question 16** ([6]). Is there (consistently) a chain transitive automorphism of \( \omega_1^* \)?

In what follows, \( \mathbb{N}^* \) is equipped with the shift map, the unique continuous extension to \( \mathbb{N}^* \) of the successor map on \( \mathbb{N} \).

**Question 17.** Is every chain transitive dynamical system of weight \( \aleph_1 \) a quotient of \( \mathbb{N}^* \)?

A “yes” is consistent by results in [7]. In the category of topological spaces, the analogous theorem, Paravočenko’s Theorem, is true in ZFC.

**Question 18.** Is it consistent that, for some topological space \( X \) and some chain transitive map \( f : X \to X \), \( X \) is a continuous image of \( \mathbb{N}^* \), but \((X, f)\) is not a quotient of \((\mathbb{N}^*, \sigma)\)?

**Question 19.** Is there a closed \( X \subseteq \mathbb{N}^* \) that meets every minimal subsystem of \( \mathbb{N}^* \) in precisely one point?

**Question 20.** Characterize the dynamical systems in which every sequence of points is proximal to an orbit.

**Algebra**

**Question 21** ([12]). Does every compact topological group have a non-(Haar-)measurable subgroup?

A “yes” is consistent with ZFC.

**Question 22** ([6]). Are the minimal right ideals homeomorphically embedded in \( \mathbb{N}^* \)? They are not. This follows from results in in [9]: some minimal ideals are weak \( P \)-sets, but others are not.

**Question 23** ([6], [22]). Is there an idempotent in \( \mathbb{N}^* \) that is both minimal and maximal with respect to the natural ordering of idempotents (\( p \leq q \) if and only if \( p + q = q + p = q \))? Again, an affirmative answer follows from results in [9]; actually, this was answered independently (and slightly earlier) by Zelenyuk in [23].

If \( S \) has uncountable cardinality and is weakly cancellative, then \( U(S) \) (the set of uniform ultrafilters on \( S \)) is a subsemigroup of \( \beta S \).

**Question 24.** Suppose \( S \) is weakly cancellative and \( |S| \) is regular. If \( p \in U(S) \) is idempotent, is there some \( q \in S^* \setminus U(S) \) such that \( p \leq_L q \)?
Miscellany

Question 25 (The Hadwigger-Nelson problem). Let $G$ be the graph obtained from $\mathbb{R}^2$ by putting an edge between two points iff they are precisely unit distance apart. What is the chromatic number of this graph?

Let’s say that a number $x$ is roundable in base $b$ if, for every $n$, there is a string of more than $n$ consecutive zeroes in the base-$b$ expansion of $x$. Note that this notion is loosely connected with approximability by rationals.

Question 26. Is there (consistently) a number that is not roundable in any base?

Question 27. Is it possible, for any $n$, to cover an $n \times n$ chessboard with $n-2$ lines? (to “cover” the board means to draw the lines in such a way that every square has some line passing through its interior)

The following is just for fun and probably not a serious research question. But I’ve thought about it quite a bit and have no idea!

Question 28. Is there a subset of $\mathbb{R}^2$ that meets every line in two (or even just finitely many) homeomorphic copies of $[0, 1]$?

References

[8] W. R. Brian and A. W. Miller, “Partitions of $2^\omega$ and completely ultrametrizable spaces,”