NON-WELL-FOUNDED EXTENSIONS OF $\forall$

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Abstract. This paper describes a new and user-friendly method for constructing models of non-well-founded set theory. Given a sufficiently well-behaved system $\theta$ of non-well-founded set-theoretic equations, we describe how to construct a model $M_\theta$ for $ZFC^-$ in which $\theta$ has a non-degenerate solution. We will prove that this $M_\theta$ is the smallest model for $ZFC^-$ which contains $\forall$ and has a non-degenerate solution of $\theta$.

1. Introduction

$ZFC^-$ is the system of axioms consisting of all the axioms of $ZFC$ except for the Axiom of Foundation. The relative consistency of $ZFC^- + \neg$Foundation has been known for some time, but only the more recent work of Scott, of Forzi and Honsell, and most notably of Aczel has revealed the rich and beautiful structure possible in a non-well-founded universe of sets. Aczel uses equivalence relations called regular bisimulations to develop a notion of equality on sets (the definition of a bisimulation is a generalization of the Axiom of Extensionality), and shows that different bisimulations lead to different models for $ZFC^-$. Barwise and Moss (see [3]) show that Aczel’s models can be viewed as extensions of the well-founded universe obtained by adding in the solutions to a fixed class of set-theoretic equations. We modify this idea in two important ways. First, we will show how to obtain non-well-founded extensions of $\forall$ by adding solutions to non-well-founded equations using a direct and transparent method. Second, we will see that this method applies not only to a handful of large fixed classes of equations determined by a bisimulation relation, but instead to any reasonably well-behaved system of equations. Given a “good” system of equations, we will show how to produce an extension of $\forall$ which contains non-degenerate solutions to those equations. These models will violate the Axiom of Foundation (hence “non-well-founded”), but without necessarily violating it in the extreme manner required by $AFA$ and similar axioms.

The construction of these models is analogous to the algebraic extension of a ring. In the case of a ring extension, the first step is to extend the base ring to a ring of polynomials. Likewise, the first step in our construction is to extend the well-founded universe $\forall$ by adding atomic objects, known as urelements, which act as indeterminates. Since the theory of urelements is semi-standard, our focus will be to explain the basic notions involved and provide some intuition and basic notation for working with urelements; the formal details can be found in the literature.

In section 3 we describe the systems of equations to which our method will apply, and in section 4 we demonstrate how each such system $\theta$ gives rise to a model $M_\theta$ for $ZFC^-$. In section 5 we prove basic results about these models, most notably that $M_\theta$ is the minimal extension of $\forall$ containing non-degenerate solutions to $\theta$. Because
the restrictions we place on our “good” systems of equations are not very severe, it
becomes easy to produce a variety of consistency and independence results, which
is the topic of Section 6.

2. URELEMENTS

The theory of urelements has its origins in Zermelo’s own formulation of set
theory in 1908 (see [11]). A more modern treatment can be found in [3] or [10]; we
mostly follow [3].

Urelements are atomic objects that we add to our set theory that can be members
of sets but that are not sets themselves. Because they are not sets, urelements do
not have any members.

This requires us to rethink the Axiom of Extensionality, which says
\[ \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \]
on, in English,

Two things are the same if they have the same elements

This is problematic because all urelements share the property of having no elements
at all. The Axiom of Extensionality, as stated, identifies any two urelements with
each other and identifies any urelement with \( \emptyset \). To get around this, we rewrite the
Axiom of Extensionality as

Two sets are the same if they have the same elements

but some things are not sets, and these have no elements.

In order to write down a formal version of this modified axiom, we need a way
of expressing formally whether or not something is a set. We accomplish this by
expanding the language of set theory by introducing a unary relation \( \sigma \) which is
interpreted intuitively as

\( \sigma(x) \leftrightarrow x \text{ is a set} \)

Our new Axiom of Extensionality then takes the form

\[ \forall x \forall y ((\sigma(x) \land \sigma(y)) \rightarrow (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)) \land \forall x (\neg \sigma(x) \rightarrow \neg \exists y (y \in x)) \]

Whereas the standard version of the Axiom of Extensionality tells us how to
distinguish between two sets in a universe of pure sets, this new axiom tells us how
to distinguish between two sets in a universe containing both sets and urelements.
All of the other axioms of ZFC still make sense in a universe containing urelements,
and we denote by ZFC' the axiom system consisting of the modified Axiom of
Extensionality and all of the other ZFC axioms in their standard form. Whereas
ZFC does not permit urelements to exist, ZFC' does, and in fact we have the
following theorem:

Theorem 1. Assume ZFC is consistent. Then there is a model of ZFC' which
contains \( \mathbb{V} \) and has a proper class of urelements.

Proof. We outline a proof; the details can be found in [3] or [10].

Define (by transfinite recursion):

\[ W_0 = \emptyset \]

\[ W_{\alpha+1} = \{0, 1\} \times P(W_\alpha) \]
\[ W_\alpha = \bigcup_{\xi \in \alpha} W_\xi \] for limit \( \alpha \)

Let \( \mathbb{W} = \bigcup_{\alpha \in \text{Ord}} W_\alpha \). Intuitively, the construction of \( \mathbb{W} \) is just a way of tagging things as either urelements or regular sets: something is an urelement if it is of the form \((0, a)\) and something is a set if it is of the form \((1, a)\).

We interpret the symbols \( \in \) and \( \sigma \) in \( \mathbb{W} \) as follows:

\[ (i, a) \in \mathbb{W} \land (j, b) \iff j = 1 \land (i, a) \in V b \]

\[ \sigma((i, a)) \iff i = 1 \]

It is relatively straightforward to check each of the axioms of \( \text{ZFC}' \) for this structure, and it is also easy to see that the class of all urelements in this structure is proper.

Note that \( V \) embeds in \( \mathbb{W} \), with a natural embedding defined by

\[ \check{x} = (1, \{ \check{y} : y \in x \}) \]

Let \( M \) be a model for \( \text{ZFC}' \) and let \( \Upsilon = \{ x : \neg \sigma(x) \} \) denote the class of urelements in \( M \). We will write \( M = V[\Upsilon] \) to emphasize that \( M \) is a universe of sets augmented by the urelements \( \Upsilon \), which for us will act as indeterminates.

The support of an object \( x \in V[\Upsilon] \) is the set of all urelements that are somehow involved in \( x \):

\[ \text{supp}(x) = TC(\{ x \}) \cap \Upsilon \]

Since whenever \( x \) is a set \( TC(\{ x \}) \) is also a set, the support of a set can never be a proper class. Given any \( J \subseteq \Upsilon \) we define

\[ V[J] = \{ x \in V[\Upsilon] : \text{supp}(x) \subseteq J \} \]

\( V[J] \) is the class of all sets that hereditarily have all of their urelements in \( J \). Alternatively, we can define \( V[J] \) hierarchically:

\[ V[J] = \bigcup_{\alpha \in \text{Ord}} U_\alpha \]

As a special case of this, notice that the class \( V[\emptyset] \) is just the class of all sets which hereditarily contain no urelements: that is, \( V[\emptyset] = V \). As (clearly) \( V[J] \subseteq V[K] \) whenever \( J \subseteq K \), \( V \subseteq V[J] \) for every \( J \subseteq \Upsilon \).

3. Systems of Equations

Let \( J \subseteq \Upsilon \). A \( J \)-system is a function

\[ \theta : J \rightarrow V[J] \]

If \( J \) is a set then any \( J \)-system is itself an element of \( V[J] \); if \( J \) is a proper class, then a \( J \)-system is a functional proper class definable from parameters in \( V[J] \). We use \( J \)-systems to encode sets (classes) of equations with indeterminates from \( J \). Intuitively, for \( x \in J \), \( x = \theta(x) \). For example, if \( J = \{ a, b \} \) and \( \theta = \{(a, \{a, b\}), (b, \{a\})\} \), we think of \( \theta \) as encoding the following system of equations:

\[ a = \{a, b\} \quad b = \{a\} \]
Our goal is to construct a model $M_\theta$ in which these equations are true, so $\theta$ is a way of naming or representing in $V[J]$ objects which we want to be in $M_\theta$.

We now proceed to put restrictions on what kinds of systems we will allow, and we attempt to justify these restrictions by examples of the pathologies we are eliminating.

First, suppose $J = \{a, b\}$ and $\theta = \{(a, \{a\}), (b, \{a\})\}$. $\theta$ encodes the system of equations given by

\[
\begin{align*}
    a &= \{a\} \\
    b &= \{a\}
\end{align*}
\]

If we were to extend $V$ such that these equations have a solution, it is clear that $a$ and $b$ should have the same value in our extension. Such redundancies are unnecessary and would make our construction only more difficult, so we require that $\theta$ be injective.

Secondly, we point out that it is useless to use $\theta$ merely to rename things. Since, ultimately, our goal is to construct an extension of $V$ in which $a$ stands in for $\theta(a)$, we gain nothing new if $\theta(a) \in V$ or if $\theta(a) \in J$. For example, take $J = \{a, b, c\}$ and

\[
\begin{align*}
    \theta(a) &= \{a\} \\
    \theta(b) &= b \\
    \theta(c) &= \emptyset, \emptyset
\end{align*}
\]

This system encodes three equations, but only the first of them is really meaningful: the others already have solutions in $V[J]$. In order to avoid such redundancies we impose the following restriction on $\theta$: for every $a \in J$, $\theta(a) \notin \Upsilon \cup V$.

Lastly, take $J = \{x\}$ and

\[
\begin{align*}
    \theta(x) &= \{x, \{x\}, \{x, \{x\}\}, \{x, \{x\}, \{x\}\}\}, \ldots
\end{align*}
\]

Because $x$ appears at every level of $\theta(x)$, it is not immediately clear whether this system should have a coherent solution. In order to avoid confusion from such pathologies we will restrict our attention to those $J$-systems which only have urelements at the top level. That is, we require that for any $x \in J$, every $y \in \theta(x)$ is either a pure set or an urelement. Such systems are called flat (see [3], pp. 67-76).

This restriction needs more justification than the other two. Although it may at first seem strong, this restriction is just a way of tidying our systems without changing their capacity for expression. In the previous example, for instance, we can perform surgery on our $J$-system to turn it into a flat system without losing any information. We first define a new variable for every element of $\theta(x)$ that is not in $V \cup J$:

\[
\begin{align*}
    y_1 &= \{x\} \\
    y_2 &= \{x, \{x\}\} \\
    y_3 &= \{x, \{x\}, \{x, \{x\}\}\} \\
    \vdots
\end{align*}
\]

We then simplify this system by making the appropriate substitutions:

\[
\begin{align*}
    y_1 &= \{x\} \\
    y_2 &= \{x, y_1\} \\
    y_3 &= \{x, y_1, y_2\} \\
    y_4 &= \{x, y_1, y_2, y_3\} \\
    \vdots
\end{align*}
\]
This new set of equations, together with the equation
\[ x = \{x, y_1, y_2, y_3, y_4, \ldots\} \]
defines a flat system over the set \( J' = \{x, y_1, y_2, y_3, y_4, \ldots\} \) which contains all of the information of our original system over \( J \). In general, this sort of surgery can always be used to turn a “deep” system into a flat one. Note, though, that in the most general case it may be necessary to use an \( \omega \)-length iteration of substitutions like the one above to make a system flat.

It is worth pointing out that a similar argument justifies our first two restrictions as well, at least in the case that \( J \) is a set. If \( \theta \) is not injective or maps urelements to pure sets, we may choose some maximal \( J' \subseteq J \) such that \( \theta \) is injective on \( J' \) and no element of \( J' \) is mapped to a pure set. The result is not necessarily a \( J' \)-system, however, since the range of \( J' \) may contain elements of \( J \setminus J' \). It is thus necessary to alter the range of \( J' \) so that it does not contain any elements of \( J \setminus J' \): do this by substituting for any \( a \in J \) either the unique \( b \in J' \) such that \( \theta(a) = \theta(b) \) or, in the case \( \theta(a) \in \mathbb{V}, \theta(a) \). Now we have a \( J' \)-system, but it may be that, after these substitutions, the restriction of \( \theta \) to \( J' \) is not injective or maps urelements to pure sets. We may continue this procedure, however, to obtain a \( J'' \)-system, a \( J''' \)-system, etc. Iterating transfinitely, one arrives at a system (in at most \( |J| \) steps) which is injective and which maps no urelement to a pure set. Thus, except perhaps in the case that \( J \) is a proper class, our restrictions on the kind of systems we will allow do not constitute substantial exclusions: we are restricting the form, not the essential content, of our systems.

Collecting these restrictions into a single definition, we say that a map \( \theta : J \to \mathbb{V}[J] \) is a good system if \( \theta \) is injective and, for each \( x \in J \), \( \theta(x) = S_x \cup J_x \), where \( S_x \) is a pure set and \( \theta \neq J_x \subseteq J \).

We now define more precisely what it means to solve a system of equations. Intuitively, a solution to a system of equations is a map with domain \( J \) which tells us how to interpret the elements of \( J \) as sets. Here we will concern ourselves with non-degenerate solutions of a system, which means that different elements of \( J \) will represent different sets.

Let \( J \subseteq \mathbb{Y} \) and let \( \theta : J \to \mathbb{V}[J] \) be a good system. Furthermore, let \( \langle M, \varepsilon \rangle \) be a model of \( ZFC^- \) with \( \mathbb{V} \subseteq M \). Let \( \varsigma : J \to M \) be a function and let \( \tilde{\varsigma} \) be the extension of \( \varsigma \) to \( \mathbb{V} \cup J \) which maps every element of \( \mathbb{V} \) to itself. We say that \( \varsigma \) is a non-degenerate solution of \( \theta \) in \( M \) if \( \varsigma \) is injective and, for each \( x \in J \), \( y \in \mathbb{V} \cup J \),
\[ \tilde{\varsigma}(y) \epsilon \varsigma(x) \iff y \in \theta(x) \]

If \( \varsigma \) is a non-degenerate solution to \( \theta \) in \( M \), then it is clear that \( \varsigma \) describes a “solution” (in the intuitive sense) to the equations represented by \( \theta \). For example, if \( J = \{a, b\} \) and \( \theta = \{\langle a, \{a\}\rangle, \langle b, \{a\}\rangle\} \), \( \theta \) represents the equations
\[ a = \{a\} \quad \text{and} \quad b = \{a, b\} \]

If \( \varsigma \) is a non-degenerate solution for \( \theta \) in \( M \), then, in \( M \)
\[ \varsigma(a) = \{\varsigma(a)\} \quad \text{and} \quad \varsigma(b) = \{\varsigma(a), \varsigma(b)\} \]

General (i.e., possibly degenerate) solutions are defined and studied in [4]; this definition, however, is significantly more complicated to state, and, as we will be interested only in non-degenerate solutions here, we content ourselves without it. The interested reader may find a more thorough study of solutions to flat systems in [2] or in [3].
4. The Construction

Let \( \theta \) be a good system over some \( J \subseteq \Upsilon \). Our goal is to construct a class \( M_\theta \) and a relation \( \varepsilon \subseteq M_\theta \times M_\theta \) having the following properties:

- \( (M_\theta, \varepsilon) \) is a model for \( ZFC^- \)
- \( M_\theta \) contains (a copy of) \( \forall \)
- \( \theta \) has a non-degenerate solution in \( M_\theta \)

The basic idea is to use the objects of \( V[J] \) as our sets in \( M_\theta \), but to expand the membership relation so that each urelement stands in for its image under \( \theta \); specifically, define \( \varepsilon \) so that if \( a \in \theta(x) \) then \( a \in x \). As stated, however, this idea needs some adjustment. In order to see the problem clearly, consider the example

\[
J = \{x, y\} \\
\theta(x) = \{x, y\} \quad \theta(y) = \{x\}
\]

It is easily checked that this defines a good system. Now consider the following members of \( V[J] \):

\[
\begin{align*}
\{x\} \\
\{\{x, \{x, y\}\}\} \\
x, \{x\}, \{x, \{x\}\}, \{\{x, \{x\}\}, \{x\}\}
\end{align*}
\]

These three expressions represent three distinct members of \( V[J] \). The problem is that, according to our system \( \theta \), we want

\[
x = \{x, y\} \quad y = \{x\}
\]

Once these identifications are made, a series of substitutions reveals each of the three expressions above to be the same; in fact, they are all just different expressions for \( y \). Thus distinct members of \( V[J] \) may represent sets that we wish to identify.

We want \( M_\theta \) to be a model for \( ZFC^- \), and in particular for the Axiom of Extensionality. If, as desired, we have \( M_\theta \models (x = \{x, y\} \land y = \{x\}) \), it would violate the Axiom of Extensionality to have, for example, \( y \) and \( \{x\} \) both as formally distinct objects of \( M_\theta \).

In order to get around this problem we will, rather than dealing with equivalence classes and quotients, simply restrict our attention to a suitable subclass \( M_\theta \subseteq V[J] \) in which these sorts of redundancies do not occur. Continuing the above example, suppose \( x, y \in M_\theta \). Since \( x \in M_\theta \) we cannot have the set \( \{x, y\} \) in \( M_\theta \), because then we would have two names for the same thing. Moreover, we have already said that we intend to define \( \varepsilon \) on \( M_\theta \) as an expansion of \( \varepsilon \). Because \( M_\theta \) must be transitive with respect to \( \varepsilon \), \( M_\theta \) will also be a transitive subclass of \( V[J] \) with respect to \( \varepsilon \).

Therefore, if we want \( \{x, y\} \notin M_\theta \), the set \( \{x, y\} \) should not appear at any level of any set in \( M_\theta \). Thus we require that if \( \{x, y\} \in TC(\{a\}) \) then \( a \notin M_\theta \). Similarly, to avoid redundant expressions for the symbol \( y \), if \( \{x\} \in TC(\{a\}) \) then \( a \notin M_\theta \).

These considerations motivate the following definition:

Let \( \theta \) be a good system over \( J \). Define

\[
M_\theta = V[J] \setminus \{y \in V[J]: (\exists x \in J)(\theta(x) \in TC(\{y\}))\}
\]

Furthermore, define \( \varepsilon \) on \( M_\theta \) by

\[
x \varepsilon y \iff x \in y \lor (y \in J \land x \in \theta(y))
\]
In English, $M_\theta$ is the class of all $x \in V[J]$ such that $TC(\{x\})$ is disjoint from the image of $\theta$, with $\varepsilon$ defined as the smallest possible expansion of $\in$ which makes $a = \theta(a)$ true for all $a \in J$.

**Lemma 1.** (Rieger’s Theorem) Let $M$ be a class and let $\varepsilon$ be a relation on $M$ which is set-like (i.e., for every $x \in M$, $\{y : y \varepsilon x\}$ is a set). If, for every subset $A$ of $M$, there is a unique $x \in M$ such that $A = \{y \in M : y \varepsilon x\}$, then $M$ is a model for $ZFC^\sim$.

**Proof.** A thorough proof of this result can be found in [9] or in Appendix B of [2]. □

**Theorem 2.** Given a good system $\theta$ over a nonempty class of urelements $J$, $\langle M_\theta, \varepsilon \rangle$ is a model for $ZFC^\sim$. Moreover, $M_\theta$ violates the Axiom of Foundation iff $J \neq \emptyset$.

**Proof.** Since $M_\theta \subseteq V[J]$ and we constructed $V[J]$ as a subclass of $V$ in Theorem 1, we may consider $M_\theta$ to be a subclass of $V$ with $\varepsilon \subseteq M_\theta \times M_\theta$. Thus Rieger’s Theorem may be applied to $\langle M_\theta, \varepsilon \rangle$. It is easy to check that $\langle M_\theta, \varepsilon \rangle$ satisfies the remaining hypotheses of Lemma 1, and it follows that $\langle M_\theta, \varepsilon \rangle$ is a model for $ZFC^\sim$.

It remains, then, to check that $\langle M_\theta, \varepsilon \rangle$ is non-well-founded when $J \neq \emptyset$. We do this by constructing an infinite descending $\varepsilon$-chain in $M_\theta$. Pick some $x_0 \in J$. By the definition of a good system there is some $x_1 \in J$ such that $x_1 \in \theta(x_0)$, i.e., such that $x_1 \varepsilon x_0$. Similarly, there is some $x_2 \in J$ such that $x_2 \varepsilon x_1$, and so forth. Proceeding this way, we obtain the desired chain:

$$\ldots \varepsilon x_3 \varepsilon x_2 \varepsilon x_1 \varepsilon x_0$$

Note that Theorem 2 is not a consistency result but a relative consistency result: in order to obtain a model for $ZFC^\sim$ we had to start with a model for $ZFC$. Gödel’s theorems tell us that this is the best we can hope for, since a model for $ZFC$ can always be produced from a model for $ZFC^\sim$.

**Theorem 3.** Given a good system $\theta$, the model $\langle M_\theta, \varepsilon \rangle$ has a non-degenerate solution for $\theta$.

**Proof.** It is clear from our definition of $M_\theta$ that $V \subseteq M_\theta$ and $J \subseteq M_\theta$. Thus we may take $\varsigma : J \rightarrow M_\theta$ to be the identity function. As $\varsigma$ obviously is injective, it remains to check that

$$(\forall x \in J)(\forall y \in V[J])(y \in \theta(x) \iff \varsigma(y) \varepsilon \varsigma(x))$$

This follows readily from the definition of $\varepsilon$, so the theorem is proved. □

5. Basic Results

Our first result tells us that, even though the models $M_\theta$ are non-well-founded, they nonetheless obey a modified form of well-founded induction and recursion.

**Lemma 2.** Let $\theta$ be a good system over some $J \subseteq \Upsilon$.

(a) Let $\phi(x)$ be a proposition with one free variable. If, for every $x \in M_\theta$,

$$\forall y \varepsilon x \phi(y) \Rightarrow \phi(x)$$

and, furthermore, $\phi(x)$ holds for every $x \in J$, then $\phi(x)$ holds for every $x \in M_\theta$. 
(b) If $F : M_\theta \times M_\theta \to M_\theta$ and $F' : J \to M_\theta$ are functional, there is a unique $G : M_\theta \to M_\theta$ such that
\[
\forall x \in J \ (G(x) = F'(x)) \quad \text{and} \quad \forall x \in M_\theta \setminus J \ (G(x) = F(x, G \upharpoonright \{ y : y \in x \}))
\]

Proof. (a) follows directly from $\in$-induction in $V[J]$ and (b) is proved from (a) by a straightforward adaptation of the usual proof of recursion. \qed

**Theorem 4.** Let $\theta$ be a good system. If $M$ is a model for ZFC$^-$ which contains (a copy of) $V$ and has a non-degenerate solution for $\theta$, then $M_\theta$ can be embedded in $M$. In other words, $M_\theta$ is the smallest universe of sets containing $V$ in which $\theta$ has a non-degenerate solution.

Proof. We have already shown how to construct $M_\theta$ inside of $V$, so we will assume that $M_\theta \subseteq M$, but that $\in$ (the membership relation for $M_\theta$) may be different from the restriction of $\in$ to $M_\theta$. We claim that there is an injective function $G$ (similar to the Mostowski collapse) with domain $M_\theta$ such that $x \in y$ if and only if $G(x) \in G(y)$.

Let $\sigma : J \to M$ be a non-degenerate solution for $\theta$. Define $G$ on $x \in J$ by $G(x) = \sigma(x)$ and then, using Lemma 2(b), define $G$ on the remainder of $M_\theta$ by
\[
G(x) = \{ G(y) : y \in x \}
\]
It follows from Lemma 2(a) that $G$ is an injective homomorphism, and so the theorem is proved. \qed

Our construction can be used to impute some very strange behavior to the membership relation. For example, consider the ordered structure $\langle \mathbb{R}, \leq \rangle$. We can force the relation $\in$ to mimic the relation $\leq$ as follows. Let $J$ be a set of urelements indexed by $\mathbb{R}$ and let $\theta$ mimic the induced order on $J$. That is, let
\[
J = \{ x_r \}_{r \in \mathbb{R}}
\]
and define $\theta$ by
\[
\theta(x_r) = \{ x_s : s \leq r \}
\]
It is easy to check that $\theta$ is a good system over $J$. Applying our construction, we obtain a model $M_\theta$ of ZFC$^-$ which contains a non-degenerate solution to $\theta$. Furthermore, $J$ is a set in $M_\theta$, and $\langle J, \in \rangle$ is isomorphic to $\langle \mathbb{R}, \leq \rangle$.

Suppose $S$ is an arbitrary set and $\mathcal{R} \subseteq S \times S$. Following the above example, consider a class of urelements $J$ indexed by $S$ and define $\theta$ on $J$ by
\[
\theta(x_s) = \{ x_t : (t, s) \in \mathcal{R} \}
\]
In general, this system may not be a good system because it may fail to be injective. In order to get around this problem, we show that any relational structure can be embedded in one that gives rise to a good system. This idea motivates the proof of the following theorem:

**Theorem 5.** Let $\mathcal{C}$ be any class and let $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$. Then there is a model $\langle M, \in \rangle$ for ZFC$^-$ and a subclass $\tilde{\mathcal{C}}$ of $M$ such that $\langle \mathcal{C}, \mathcal{R} \rangle$ is isomorphic to $\langle \tilde{\mathcal{C}}, \in \rangle$.

Proof. Let $J$ be a set of urelements indexed by $\mathcal{C} \times \{ 0, 1, 2 \}$:
\[
J = \{ x_{(s,i)} \}_{s \in \mathcal{C}, i \in \{ 0, 1, 2 \}}
\]
Define $\theta$ on $J$ by
\[
\theta(x_{(s,0)}) = \{ x_{(t,0)} : (s, t) \in \mathcal{R} \} \cup \{ x_{(s,1)}, x_{(s,2)} \}
\]
and \( \theta(x_{j,i}) = \{ x_{j,i} \} \) for \( i = 1, 2 \)

It is clear that \( \theta \) is a flat system over \( J \) and that, for each \( s \in J \), \( \theta(s) \) contains some member of \( J \). Furthermore, \( \theta \) is injective since each \( x_{(s,1)} \) and \( x_{(s,2)} \) maps to its own singleton and each \( x_{(s,0)} \) maps to some set containing both \( x_{(s,1)} \) and \( x_{(s,2)} \), and no other urelements of the form \( x_{(r,i)} \) for \( i = 1, 2 \). Therefore \( \theta \) is a good system. Applying Theorem 2, we obtain a model \( M_\theta \) for \( ZFC^- \). Let

\[ \tilde{C} = \{ x_{(s,0)} \} \in C \subseteq J \]

It is easy to check that \( \tilde{C} \subseteq M_\theta \) and that \( (\tilde{C}, \varepsilon) \) is isomorphic to \( (C, R) \).

To paraphrase the above theorem, any relation can be mimicked in some model for \( ZFC^- \). If we restrict our attention to relations on sets (rather than on arbitrary classes), we can strengthen this result by reversing the quantifiers: there is some fixed model for \( ZFC^- \) in which every relation over a set can be mimicked. Such a model satisfies what has come to be known as Boffa’s Anti-Foundation Axiom or \( BAFA \). The proof of this essentially involves collecting all set relations into a class and then invoking Theorem 5. For more on \( BAFA \) and another approach to modeling it, see [2], pp. 57-69.

6. INDEPENDENCE RESULTS

The construction explained in §4 does not give just one new model for \( ZFC^- \), but a general method for producing an infinite variety of new models. Simply by varying the system \( \theta \) we can manipulate the properties of the derived model \( M_\theta \). Furthermore, the method is “user-friendly” in that it is often easy to guess many of the properties of \( M_\theta \) from \( \theta \). In this way we are able to obtain a variety of independence results, and a representative sample of these will be the topic of this section.

For our first example we will show how to construct a model for \( ZFC^- \) containing more than one Quine atom, that is, more than one set satisfying the relation \( x = \{ x \} \).

Let \( J \) be any non-empty class of urelements. Define \( \theta \) over \( J \) by

\[ \theta(x) = \{ x \} \quad \text{for all } x \in J \]

\( \theta \) is easily seen to be a good system. Applying Theorem 2, \( M_\theta \) is a model for \( ZFC^- \). If \( x \in J \) then \( x \in M_\theta \) and, by the definition of \( \varepsilon \),

\[ (\forall z \in M_\theta)(z \varepsilon x \iff z = x) \]

So each element of \( J \) represents a Quine atom in \( M_\theta \). Since there were no size restrictions on \( J \), this shows that a non-well-founded universe of sets may have any finite number of Quine atoms, an infinite set (of any given cardinality) of Quine atoms, or a proper class of Quine atoms.

Next consider the following propositions:

(1) There is a unique Quine Atom
(2) There is a Quine Atom
(3) There is a reflexive set
(4) There is a circular set
(5) The Axiom of Foundation is false

(A reflexive set is a set \( x \) such that \( x \in x \), and a circular set is a finite set \( \{ x_0, x_1, \ldots, x_n \} \) such that \( x_0 \in x_1 \in \ldots \in x_n \in x_0 \).)

\( \square \)
Theorem 6. Each of (1) – (5) is consistent with and independent of ZFC$^{-}$. Furthermore, (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5), but (5) $\not\Rightarrow$ (4) $\not\Rightarrow$ (3) $\not\Rightarrow$ (2) $\not\Rightarrow$ (1).

Proof. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Furthermore, the above comments make it clear that (1) is consistent, and it follows that each of (1) – (5) is consistent. In order to prove that none of the reverse implications holds, we will need to construct four different models of ZFC$^{-}$: one in which (5) holds but (4) does not, one in which (4) holds but (3) does not, etc.

(5) $\not\Rightarrow$ (4): Take $J$ to be a countable set, $J = \{x_0, x_1, x_2, \ldots\}$, with

$$\theta(x_i) = \{x_{i+1}\} \quad \text{for every } i \in \mathbb{N}$$

By Theorem 2, $M_\theta$ is a model of ZFC$^{-}$; it remains to show that $M_\theta$ contains no circular set. We will prove the slightly stronger fact that every finite subset of $M_\theta$ has an initial element.\(^1\) Let $Y = \{y_0, y_1, \ldots, y_n\} \subseteq M_\theta$. If $Y$ contains none of the $x_i$ then $\varepsilon$ is just the restriction of $\in$ to $Y$, and so $\varepsilon$ is well-founded on $Y$ since $\varepsilon$ is. If $Y$ does contain one of the $x_i$ then it must contain one of maximal index, and it is clear from the definition of $\varepsilon$ that this set has no members in $Y$. In either case $Y$ has an initial element.

(4) $\not\Rightarrow$ (3): Fix some finite number $n > 0$ and let $J = \{x_0, x_1, \ldots, x_n\}$. Define $\theta$ by

$$\theta(x_n) = \{x_{n-1}\}, \quad \theta(x_{n-1}) = \{x_{n-2}\}, \ldots \quad \theta(x_1) = \{x_0\}, \quad \theta(x_0) = \{x_n\}$$

It is clear that $\theta$ is a good system and hence gives a model $M_\theta$ for ZFC$^{-}$, and that $\{x_0, x_1, \ldots, x_n\}$ is a circular set in $M_\theta$. To prove that there is no $x \in M_\theta$ such that $x \in x$, note that this is true for all $x \in J$ by the definition of $\varepsilon$ and then apply Lemma 2(a) to show that it holds for every $x \in M_\theta$. Notice that this model is easily modified to show that we can have a unique circular set of size $n$ for any $n$ or multiple circular sets of any desired sizes.

(3) $\not\Rightarrow$ (2): Let $J$ contain a single element $x$ and take

$$\theta(x) = \{\emptyset, x\}$$

Clearly $\theta$ gives a good system over $J$ and hence a model of ZFC$^{-}$. Since $x \in x$, $M_\theta$ contains a reflexive set. It follows easily from Lemma 2(a) that $x$ is the only reflexive set in $M_\theta$. Since $x$ is not a Quine atom ($x \neq \emptyset \in x$), this completes the proof.

(2) $\not\Rightarrow$ (1): We have already mentioned how to construct a model for ZFC$^{-}$ in which there are multiple Quine atoms. \(\square\)

In what follows we use $S$ to denote the successor function $x \mapsto x \cup \{x\}$.

Theorem 7. The following propositions are consistent with and independent of ZFC$^{-}$:

(1): The successor function is injective
(2): $S(x) \in x \iff x \in x$
(3): $\forall x (\emptyset \in TC(\{x\}))$
(4): If $x$ is transitive then $x \setminus \{x\} \notin x$

Proof.

\(^1\)Recall that $x$ is an initial element of $X$ if, for every $y \in x$, $y \notin X$. Note that a circular set is a finite set with no initial elements. The Axiom of Foundation, in its usual form, is simply the statement that every set has an initial element.
(3): This proposition fails if there is a Quine atom \( x = \{ x \} \), since \( \emptyset \notin x = TC(\{ x \}) \), and we have already seen that the existence of Quine atoms is consistent with \( ZFC^- \). On the other hand, (3) follows from the Axiom of Foundation. To see that (3) is also consistent with \( ZFC^- + \neg \text{Foundation} \), consider the good system over \( J = \{ x \} \) defined by 

\[ \theta(x) = \{ x, \emptyset \} \]

It is an easy consequence of Lemma 2(a) that \( \emptyset \in TC(\{ x \}) \) for every \( x \in M_\theta \).

(1),(2),(4): These proofs are similar in style to that of (1) and the details are omitted.

\[ \square \]

References


