Preserving topological properties under refinement

Will Brian

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Proposition (folklore)

If \( A \) is a subset of a topological space, then the following are equivalent:

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- $A$ is the intersection of an open set and a closed set.
- $A = U - V$, with $U$ and $V$ both open (or both closed).

For every $x \in A$, there is a neighborhood of $x$ on which $A$ is a closed set.

Sets of this kind are called locally closed.
Locally closed sets

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Lower separation properties

Proposition (easy)

If a topology \( \sigma \) is \( T_0 \), \( T_1 \), \( T_2 \), or \( T_{2\frac{1}{2}} \), then every refinement of \( \sigma \) also has this property.
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Theorem

Let $\sigma$ be any topology on a set $X$ and let $A \subseteq X$. If $A$ is not locally closed, then $\langle \sigma, A \rangle$ is not $T_3$. If $\sigma$ is $T_3$ then the converse also holds: $\langle \sigma, A \rangle$ is $T_3$ if and only if $A$ is locally closed.
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### Corollary

Every topology in $[\sigma, \tau]$ is $T_3$ if and only if $\sigma$ is $T_3$ and every member of $\tau - \sigma$ is locally closed (with respect to $\sigma$).
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The obvious examples of spaces like this are the scattered spaces of Cantor-Bendixson rank 1 or 2. These are not the only examples of submaximal spaces (Hewitt, van Douwen, and Alas et al. have given others). Nonetheless, they are the only “concrete” examples (any non-scattered example contains the base of an ultrafilter in its topology).
The exact same results hold for $T_{3\frac{1}{2}}$ spaces:

**Theorem**

- If $\sigma$ is $T_{3\frac{1}{2}}$ then $\langle \sigma, A \rangle$ is $T_{3\frac{1}{2}}$ if and only if $A$ is locally closed.
- If $\sigma$ is $T_{3\frac{1}{2}}$ then every member of $[\sigma, \tau]$ is $T_{3\frac{1}{2}}$ if and only if every member of $\tau - \sigma$ is locally closed.
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- A $T_{3\frac{1}{2}}$ space is submaximal if and only if every refinement of it is also $T_{3\frac{1}{2}}$.

The same is not true for the $T_4$ property. In fact, there is a $T_4$ topology $\sigma$ on a set $X$ and a point $x \in X$ such that $\langle \sigma, \{x\} \rangle$ is not $T_4$. 

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If \( \sigma \) is a (completely) metrizable topology, then \( \langle \sigma, A \rangle \) is (completely) metrizable if and only if \( A \) is locally closed.
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For metrizability, use the Nagata-Smirnov metrization theorem. For completeness, use the notion of completeness provided by the strong Choquet game. If II has a winning strategy in both \( \sigma \) and \( A \), then we can get a winning strategy for II in \( \langle \sigma, A \rangle \). (In fact, we can prove that \( \langle \sigma, A \rangle \) is strong Choquet if and only if \( A \) is – this property is implied by \( A \) being \( G_\delta \), so locally closed sets don’t work for everything!)
Some corollaries

Corollary

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Proposition

If \( \sigma \) is zero-dimensional, then \( \langle \sigma, A \rangle \) is zero-dimensional if and only if \( A \) is locally closed.
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**Corollary**

If $\sigma$ is (completely) ultrametrizable, then $\langle \sigma, A \rangle$ is (completely) ultrametrizable if and only if $A$ is locally closed.
The analogy with the $T_3$ and $T_{3\frac{1}{2}}$ properties breaks down after this point for metrizability: there is no good analogue of the results about arbitrary refinements of $T_3$ ($T_{3\frac{1}{2}}$) spaces and submaximality.

**Proposition**

*Every non-discrete topology has a non-metric refinement.*

However, the zero-dimensionality property does give a result of this kind:

**Proposition**

*A zero-dimensional space is submaximal if and only if every refinement of it is also zero-dimensional.*
Small inductive dimension

We write $\text{ind}_\sigma(X)$ for the small inductive dimension of $X$ when it is given the topology $\sigma$. 

Theorem 1
If $\sigma$ is a regular topology on $X$ and $A \subseteq X$ is locally closed with respect to $\sigma$, then $\text{ind}_{\langle \sigma, A \rangle}(X) \leq \text{ind}_\sigma(X)$.

Theorem 2
If $\sigma$ is a metrizable, locally compact topology on $X$ and $A \subseteq X$ is locally closed with respect to $\sigma$, then $\text{ind}_{\langle \sigma, A \rangle}(X) = \text{ind}_\sigma(X)$.

Question
Can this be improved to equality in the general case?
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1. If $\sigma$ is a regular topology on $X$ and $A \subseteq X$ is locally closed with respect to $\sigma$, then $\text{ind}_{(\sigma, A)}(X) \leq \text{ind}_\sigma(X)$.
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*Can this be improved to equality in the general case?*
More on the small inductive dimension

While the implication reverses for dimension 0, it does not reverse in general:

Proposition

If $\sigma$ is the usual topology on $\mathbb{R}$ and $A \subseteq \mathbb{R}$, then $\text{ind}_{\langle \sigma, A \rangle}(\mathbb{R}) = 1$. 

In contrast to this, we also have

Proposition

For each $n$, there is a (compact, metrizable) topology $\sigma$ on a set $X$ and some $A \subseteq X$ such that $\text{ind}_{\sigma}(X) = n$ and $\text{ind}_{\langle \sigma, A \rangle}(X) = n + 1$. 

Proposition

If $\sigma$ is any topology on $X$ and $A \subseteq X$, then $\text{ind}_{\langle \sigma, A \rangle}(X) \leq \text{ind}_{\sigma}(X) + 1$. 

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Local compactness

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**Theorem**

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*If $\sigma$ is locally compact, then $\langle \sigma, A \rangle$ is locally compact if and only if both $A$ and its complement are locally closed.*

**Proposition**

*For each locally compact topology $\tau$ on $X$, there is a compact topology $\sigma$ on $X$ and some $A \subseteq X$ such that $\tau = \langle \sigma, A \rangle$.***
Thank you for listening!