A CARDINAL INVARIANT RELATED TO CLEAVABILITY

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Abstract. A space $X$ is $\kappa$-cleavable over $Y$ if, for any partition of $X$ into $\kappa$ disjoint sets, there is a continuous function $f : X \to Y$ such that the images of these sets under $f$ are pairwise disjoint. This notion defines a cardinal function on $Y$, namely the least $\kappa$ such that whenever $X$ is $\kappa$-cleavable over $Y$ then there is a continuous injection $X \to Y$. After a brief exploration of $\kappa$-cleavability in general, we investigate $\kappa$-cleavability over $\mathbb{R}^2$. We prove that a $\sigma$-compact polyhedron $X$ is 6-cleavable over $\mathbb{R}^2$ if and only if $X$ embeds in $\mathbb{R}^2$.

1. Introduction

If $X$ and $Y$ are topological spaces, we say that $X$ is cleavable over $Y$ if and only if, for any subset $A \subseteq X$, there is a continuous function $f : X \to Y$ such that $f[A] \cap f[X \setminus A] = \emptyset$. Intuitively, the image of $A$ under $f$ is a “reflection” of $A$ in $Y$, i.e., a near-copy of $A$ which sits inside of a different, possibly simpler space. This intuition is supported by the fact that there are many theorems of the form “if $Y$ has property $P$ and $X$ is cleavable over $Y$, then $X$ has property $P$”; see [1] for examples.

We can re-interpret the definition of cleavability as talking about partitions: $X$ is cleavable over $Y$ if and only if, for any partition of $X$ into two disjoint sets $A$ and $B$, there is a continuous function $f : X \to Y$ such that $f[A] \cap f[B] = \emptyset$. Generalizing to partitions of arbitrary size, we say that $X$ is $\kappa$-cleavable over $Y$ if and only if, for any partition $\mathcal{P}$ of $X$ into $\kappa$ disjoint sets, there is a continuous function $f : X \to Y$ such that the images of these sets under $f$ are pairwise disjoint. In this case we say that $f$ respects the partition $\mathcal{P}$. If $\lambda < \kappa$ then $\kappa$-cleavability implies $\lambda$-cleavability, and, if $\kappa$ is the least cardinal such that $X$ is not $\kappa$-cleavable over $Y$, we say that $\kappa$ is the cleavability of the pair $(X,Y)$ and write $\kappa = \clubsuit(X,Y)$. If $X$ is $\kappa$-cleavable over $Y$ for every cardinal $\kappa$, we write $\heartsuit(X,Y) = \infty$. Note that if $Y$ is nonempty then $\heartsuit(X,Y) \geq 2$ for any $X$. 

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Proposition 1.

(i) \( \clubsuit(X,Y) = \infty \) if and only if \( X \) condenses into \( Y \) (i.e., there is a continuous injection \( X \to Y \)).

(ii) If \( \clubsuit(X,Y) \neq \infty \) then \( \clubsuit(X,Y) \leq |Y|^+ \).

(iii) For any space \( Y \) and any class \( \mathcal{C} \) of topological spaces, there is a cardinal \( \kappa \) such that whenever \( X \in \mathcal{C} \) is \( \kappa \)-cleavable over \( Y \), \( X \) condenses into \( Y \).

Proof.

(i) If \( \clubsuit(X,Y) = \infty \) then \( X \) is \(|X|^+\)-cleavable over \( Y \). Consider the partition of \( X \) into singletons: any continuous function which respects this partition is an injection.

(ii) Suppose \( X \) is \(|Y|^+\)-cleavable over \( Y \). If \( |X| \leq |Y| \), then the argument from (i) proves that \( X \) condenses into \( Y \), contrary to assumption. Thus \( |X| > |Y| \) and we may partition \( X \) into \(|Y|^+\) nonempty subsets; the definition of \(|Y|^+\)-cleavability then implies that there are at least \(|Y|^+\) disjoint nonempty subsets of \( Y \).

(iii) This follows from (i) and (ii).

Part (iii) of this proposition defines a class of cardinal functions on \( Y \): for any class \( \mathcal{C} \) of topological spaces, let \( \clubsuit_\mathcal{C}(Y) \) denote the least cardinal \( \kappa \) such that if \( X \in \mathcal{C} \) is \( \kappa \)-cleavable over \( Y \) then \( X \) condenses into \( Y \). We will say that \( \clubsuit_\mathcal{C}(Y) \) is the cleavage of \( Y \) relative to the class \( \mathcal{C} \).

For any space \( Y \), the discrete space on \(|Y|^+\) points is \(|Y|^+\)-cleavable over \( Y \), yet there is no continuous injection from this space into \( Y \). Thus the notion of cleavage becomes trivial if we allow the class \( \mathcal{C} \) to be too large. Nonetheless, there are many examples of classes \( \mathcal{C} \) and spaces \( Y \) for which \( \clubsuit_\mathcal{C}(Y) \) takes more interesting values. For example, Arhangel’skii proved that the cleavage of \( \mathbb{R} \) relative to compact Hausdorff spaces is 2 (see [2], Theorem 9). Buzyakova proved that the cleavage of any LOTS relative to compact continua is also 2 (see [3], Theorem 4.2). Motivated by these examples, we will prove in section 3 that the cleavage of \( \mathbb{R}^2 \) relative to the class of \( \sigma \)-compact polyhedra is 6.

The notion of \( \kappa \)-cleavability was suggested to the author in a passing remark by Levine, to whom also is due our \( \clubsuit \) notation. Nonetheless this paper is, as far as we know, the literary debut of both \( \kappa \)-cleavability and cleavage (the latter concept being the author’s own). As it is impossible to explore every avenue of inquiry for a new idea in a single paper, many open questions for the interested reader will be articulated along the way.
2. Nontriviality

In this section and throughout this paper we will have occasion to talk about graphs. Although graphs can be considered purely combinatorial objects, for us they will be topological. If \( G \) is a graph with vertex set \( V \) and edge set \( E \) then we will consider \( G \) to be the topological space obtained from \( |E| \) disjoint copies of \([0,1]\) by an appropriate identification of endpoints.

The following theorem shows that the notion of \( \kappa \)-cleavability is nontrivial, i.e., strictly stronger than the notion of cleavability.

**Theorem 2.** For every cardinal \( \mu \geq 2 \), there is a pair \((X,Y)\) of Hausdorff spaces such that \( \clubsuit(X,Y) = \mu \).

**Proof.** For any cardinal \( \mu \), let \( K_\mu \) denote the complete graph on \( \mu \) points, i.e., a graph on \( \mu \) points in which a single edge joins every pair of points. For any cardinal \( \lambda \), let \( K^\mu_\lambda \) denote the complete graph on \( \lambda \) points, but with \( \mu \) edges adjoining every pair of vertices and \( \mu \) edges adjoining each vertex to itself (thus \( K^\mu_\lambda = K^\lambda_\mu \)).

Let \( \mu \) be an infinite cardinal. Let \( X = K_\mu \) and let \( Y = \bigsqcup_{\lambda<\mu} K^\mu_\lambda \); the topology on \( Y \) is the free union, i.e., each \( K^\mu_\lambda \) is clopen in \( Y \). We show that \( \clubsuit(X,Y) = \mu \).

First we must argue that, for any \( \lambda < \mu \), \( X \) is \( \lambda \)-cleavable over \( Y \). Let \( \lambda < \mu \) and let \( \{A_\alpha\}_{\alpha<\lambda} \) be a partition of \( X \). We will show that there is a continuous map \( f : X \to K^\mu_\lambda \subseteq Y \) such that \( f[A_\alpha] \cap f[A_\beta] = \emptyset \) for distinct \( \alpha, \beta < \lambda \). For simplicity, we will write the set of vertices of \( K^\mu_\lambda \) as \( V = \{v_\alpha\}_{\alpha<\lambda} \) and we will use the symbols \( w, w', \) etc. when referring to vertices of \( X \). If \( w \) is a vertex of \( X \) and \( w \in A_\alpha \), we let \( f \) map \( w \) to \( v_\alpha \). If \( E \) is the edge of \( X \) which connects \( w \) and \( w' \), and if \( w \in A_\alpha, w' \in A_\beta \), then we extend \( f \) by mapping \( E \) onto one of the \( \mu \) edges connecting \( v_\alpha \) and \( v_\beta \); we do this in such a way that \( f \) restricts to a homeomorphism on \( E \) and such that distinct edges of \( X \) map to distinct edges of \( K^\mu_\lambda \). The resulting map \( f : X \to K^\mu_\lambda \) is a witness to the fact that \( X \) is \( \lambda \)-cleavable over \( Y \): \( f \) is injective off of the vertices of \( X \), and it is clear that no two vertices in different partition elements map to the same point of \( K^\mu_\lambda \).

Next we show that \( X \) is not \( \mu \)-cleavable over \( Y \). We partition \( X \) by, for every vertex \( w \) of \( X \), declaring \( \{w\} \) to be an equivalence class of our partition and declaring each edge (without the endpoints) to constitute an equivalence class of our partition. Suppose that \( f : X \to Y \) is a continuous function which respects this partition, i.e., which maps points in different equivalence classes to different points. Since \( f \) is continuous and \( X \) is connected, \( f[X] \) is a connected subset of \( Y \) and
hence $f[X] \subseteq K_\lambda^\mu$ for some $\lambda < \mu$. Now suppose that $f$ maps one of the vertices of $X$ to some point that is not a vertex of $K_\lambda^\mu$; it is clear from the continuity of $f$ that some two edges of $X$ will have overlapping images, contradicting the fact that each edge of $X$ constitutes an equivalence class of our partition. Thus the vertices of $X$ map to the vertices of $K_\lambda^\mu$. Since there are only $\lambda$ vertices of $K_\lambda^\mu$, some two vertices of $X$ must map to the same point, contradicting the fact that each vertex of $X$ constitutes an equivalence class of our partition.

This proves the theorem for infinite cardinals, and it remains to prove it for finite cardinals. A straightforward modification of the above argument suffices, and we leave the details to the reader. Alternatively, there is a second approach to proving the finite case, one that does not readily generalize to infinite cardinals, and we will see the details of this second approach in the proof of Theorem 4.

Lemma 3. Let $X$ and $Y$ be topological spaces.
(i) If $A \subseteq X$ then $\clubsuit(A,Y) \geq \clubsuit(X,Y)$.
(ii) If $A \subseteq Y$ then $\clubsuit(X,A) \leq \clubsuit(X,Y)$.
(iii) If $\hat{X}$ is coarser than $X$ then $\clubsuit(\hat{X},Y) \leq \clubsuit(X,Y)$.
(iv) If $\hat{Y}$ is coarser than $Y$ then $\clubsuit(X,\hat{Y}) \geq \clubsuit(X,Y)$.

Proof. Straightforward. □

The next theorem shows that the notion of cleavage is also nontrivial, and indeed can take on any finite value greater than 2.

Theorem 4. Let $\mathcal{C}$ be the class of compact spaces. For every $n > 0$, $\clubsuit_{\mathcal{C}}(\omega \cdot n + 1) = n + 1$. Thus every finite $n \geq 2$ is represented as the cleavage of some space relative to the class of compact spaces.

Proof. Fix $Y = \omega \cdot n + 1$. This theorem is proved in two parts: first we must find a compact space which is $n$-cleavable over $Y$ but which does not embed in it, and then we must prove that every compact space which is $(n + 1)$-cleavable over $Y$ embeds in $Y$.

Let $X = \omega \cdot (n+1)+1$ and let $\mathcal{P}$ be any partition of $X$ into $n$ subsets. $X$ has $n + 1$ limit points, so there is some element of $\mathcal{P}$ containing two distinct limit points of $X$; without loss of generality, suppose $\omega$ and $\omega \cdot 2$ are in the same $\mathcal{P}$-equivalence class. Define $f : X \to Y$ by

$$f(\alpha) = \begin{cases} 
  n \cdot 2 & \text{if } \alpha = n < \omega \\
  n \cdot 2 - 1 & \text{if } \alpha = \omega + n \text{ for some } 0 \neq n < \omega \\
  \omega & \text{if } \alpha = \omega \text{ or } \alpha = \omega \cdot 2 \\
  \omega \cdot (m-1) + n & \text{if } \alpha = \omega \cdot m + n > \omega \cdot 2 
\end{cases}$$

Clearly, $f$ is a continuous function which respects the partition $\mathcal{P}$. Thus $X$ is $n$-cleavable over $Y$. This proves that $\clubsuit(X,Y) \geq n + 1$. □
We now show that $X$ is not $(n+1)$-cleavable over $Y$. The following argument is essentially due to Levine (see [10], Theorem 3.1) but we reproduce it here for completeness. Let $\mathcal{P}$ be the following $(n+1)$-partition of $X$ (here, following standard notational conventions, we identify an ordinal number with the set of its predecessors):

$$\mathcal{P} = \{\omega \cdot 2 \setminus \omega, \omega \cdot 3 \setminus \omega \cdot 2, ..., \omega \cdot (n+1) \setminus \omega \cdot n, \omega \cup \{\omega \cdot (n+1)\}\}$$

Suppose $f : X \to Y$ is a continuous function which respects this partition. Notice that every neighborhood of every limit point of $x \in X$ contains points of a different equivalence class than that of $x$; this implies that $f$ cannot map any limit point of $X$ to an isolated point. But then there are only $n$ possible values of $f$ for the $n+1$ different limit points of $X$, and no two of them belong to the same equivalence class; thus $f$ cannot respect this partition. This proves that $\bullet(X,Y) \leq n+1$.

Now let $K$ be any compact space which is $(n+1)$-cleavable over $Y$. Then $K$ is cleavable over $Y$ and, by a recent result of Levine (see [10], Theorem 2.12), any compact space which is cleavable over a countable ordinal is homeomorphic to a countable ordinal. By the argument of the previous paragraph together with Lemma 3, $K$ is homeomorphic to some compact (i.e., non-limit) ordinal $\alpha < \omega \cdot (n+1) + 1$. Every such ordinal is homeomorphic to a subset of $\omega \cdot n + 1$.

**Question 5.** For a given infinite cardinal $\kappa$, is there a space whose cleavage relative to compact spaces is $\kappa$? Can such a space be Hausdorff or compact Hausdorff?

### 3. The Cleavage of $\mathbb{R}^2$

Recall that a CW-complex is constructed inductively: its 0-skeleton is a discrete set of points, its 1-skeleton is obtained by gluing copies of $[0,1] = I$ onto its 0-skeleton, its 2-skeleton is obtained by gluing copies of $I^2$ onto its 1-skeleton, etc. Every CW-complex has a cellular structure, but, as with graphs, we will not be so concerned with the combinatorial structure of these spaces as with their topology. Following standard practice, we use the term polyhedron to refer to a finite-dimensional CW-complex considered as a topological space.

We do not give a precise definition of CW-complexes here. Although we prove several theorems below concerning polyhedra, we will not need to use their definition in these proofs: it will be enough to cite theorems from the literature. The uninitiated reader is referred to the appendix of [7] for a precise definition of CW-complexes and a survey of their topological properties.
In what follows we will denote the class of \(\sigma\)-compact polyhedra by \(\sigma CP\).

Recall that \(K_{3,3}\) is the graph on two groups of 3 vertices, with two vertices connected if and only if they belong to different groups (see picture below). Notice that \(K_{3,3}\) is a compact 1-dimensional polyhedron: it is a set of 6 vertices to which copies of \([0, 1]\) have been glued in the appropriate way. Likewise \(K_5\), the complete graph on five points, is also a compact 1-dimensional polyhedron.

Kuratowski’s Theorem, a fundamental result in graph theory, states that a finite graph \(G\) embeds in \(\mathbb{R}^2\) if and only if neither \(K_{3,3}\) nor \(K_5\) embeds in \(G\) (see [5], Theorem 11.13).

**Lemma 6.** \(\clubsuit(K_5, \mathbb{R}^2) = 5\) and \(\clubsuit(K_{3,3}, \mathbb{R}^2) = 6\).

**Proof.** First we show that \(K_5\) is not 5-cleavable over \(\mathbb{R}^2\). To do this we must find a 5-partition of \(K_5\) which is not respected by any continuous function \(f: K_5 \to \mathbb{R}^2\). Let the vertices of \(K_5\) be denoted \(v_1, v_2, v_3, v_4, v_5\) and let the edge connecting \(v_i\) to \(v_j\) (not including endpoints) be denoted \(E_{i,j}\). We partition \(X\) into the following 5 sets:

\[
E_{1,2} \cup E_{1,4} \cup \{v_1\}, \quad E_{2,3} \cup E_{2,5} \cup \{v_2\}, \quad E_{3,4} \cup E_{1,3} \cup \{v_3\}, \\
E_{4,5} \cup E_{2,4} \cup \{v_4\}, \quad E_{1,5} \cup E_{4,5} \cup \{v_5\}
\]

Suppose \(f\) is a continuous map which respects this partition. Then \(f\) maps each of the five vertices of \(K_5\) to distinct points of \(\mathbb{R}^2\). For each pair \(v_i, v_j\), the set \(f[E_{i,j} \cup \{v_i, v_j\}]\), although it may not itself be homeomorphic to \(I\) (e.g., space-filling curves), nonetheless contains a homeomorphic copy of \(I\), with \(v_i\) and \(v_j\) acting as the endpoints (this is a theorem of Hahn and Mazurkiewicz; see [14], Theorem 31.5). Thus the image of \(K_5\) under \(f\) contains a “picture” of \(K_5\) (more precisely, \(f[K_5]\) contains a union of 10 closed intervals, possibly overlapping, with endpoints identified in the appropriate way). Let \(g\) be a map from \(K_5\) onto this “picture” of \(K_5\). It is clear that we may choose \(g\) so that its restriction to any \(E_{i,j}\) is a homeomorphism, so that \(g[K_5] \subseteq f[K_5]\), and so that, for any element \(A\) of our partition, \(g[A] \subseteq f[A]\). In particular, \(g\) respects our partition if \(f\) does.

By Kuratowski’s Theorem, \(g\) cannot be an embedding. As \(g\) is continuous and \(K_5\) is compact, \(g\) must not be injective. As \(g\) is injective on vertices and on individual edges, \(g\) must map some two points of different edges to the same place. If \(g\) mapped only points from adjacent edges to the same place then we could modify \(g\) to obtain an embedding of \(K_5\). Thus \(g\) must map two points from non-adjacent edges to the same place. Since no two points from non-adjacent edges are in the
same equivalence class, $g$ cannot respect our partition, and therefore neither can $f$. Thus $K_5$ is not 5-cleavable over $\mathbb{R}^2$, i.e., $\clubsuit(K_5, \mathbb{R}^2) \leq 5$.

The proof that $\clubsuit(K_{3,3}, \mathbb{R}^2) \leq 6$ is similar.

Next we show that $K_5$ is 4-cleavable over $\mathbb{R}^2$. Since $K_5$ has 5 vertices, any 4-partition of $K_5$ must have two vertices in the same equivalence class. The following picture shows how to obtain a continuous map $f : K_5 \to \mathbb{R}^2$ which identifies two vertices of $K_5$ but is otherwise injective; this map clearly can be used to respect any given 4-partition of $K_5$. Thus $\clubsuit(K_5, \mathbb{R}^2) \geq 5$. For $K_{3,3}$ there are, up to homeomorphism, two distinct ways to identify two distinct vertices. Both of these are shown below, and, as is apparent, each results in a planar graph, indicating by a similar argument that $\clubsuit(K_{3,3}, \mathbb{R}^2) \geq 6$.

\begin{proof}
If $G$ is a finite graph then $\clubsuit(G, \mathbb{R}^2) = 5$, $\clubsuit(G, \mathbb{R}^2) = 6$, or $\clubsuit(G, \mathbb{R}^2) = \infty$.

\textbf{Proof.} If $G$ is a finite graph then by Kuratowski’s Theorem either $G$ embeds in $\mathbb{R}^2$ or $G$ contains a homeomorphic copy of $K_5$ or $K_{3,3}$. The same, in fact, is true of countable graphs, by an extension of this result likely due to Erdős (see [4] or [13]). If $G$ can be embedded in $\mathbb{R}^2$ then $\clubsuit(G, \mathbb{R}^2) = \infty$. If $G$ contains a copy of $K_5$ or of $K_{3,3}$ then, by Lemmas 3 and 6, $\clubsuit(G, \mathbb{R}^2) \leq 6$. Thus it suffices to show that any countable graph $G$ is 4-cleavable over $\mathbb{R}^2$. Given a 4-partition of $G$, we can map
\end{proof}
each of the vertices of $G$ to one of the four vertices in the picture shown below (while respecting our partition on vertices), and we can then map in the edges injectively as necessary:

Corollary 8. If $G$ is an arbitrary graph with edge set $E$ and $|E| \leq c$, then $\clubsuit(G, \mathbb{R}^2) > 4$.

Proof. A variation of the above picture suffices for this proof. □

We now turn our attention from 1-dimensional polyhedra (graphs) to $\sigma$-compact polyhedra.

Lemma 9 (Arhangel’skiǐ). Let $X$ be a separable space cleavable over $\mathbb{R}^n$. Then every subspace $U$ of $X$ that is homeomorphic to $\mathbb{R}^n$ is open in $X$.

Proof. See [2], Theorem 6. □

Let the $Q$-space be the topological space obtained from $I = [0, 1]$ and $I^2$ by gluing an endpoint of $I$ to the middle of $I^2$; this space is sometimes also called the disk with feeler, the disk with sticker, or $F_2$.

Lemma 10. Neither $S^2$ nor the $Q$-space is cleavable over $\mathbb{R}^2$.

Proof. For the $Q$-space this follows directly from Lemma 9. For $S^2$, we recall the Borsuk-Ulam Theorem ([7], Theorem 1.10): every continuous function $S^2 \rightarrow \mathbb{R}^2$ maps some two antipodal points to the same place. It follows that, if $\mathcal{P}$ is any 2-partition of $S^2$ which separates into different equivalence classes every pair of antipodal points, there is no continuous function $S^2 \rightarrow \mathbb{R}^2$ that respects $\mathcal{P}$. □
Lemma 11 (Márquez, Ayala, and Quintero). Let \( X \in \sigma CP \). Then \( X \) embeds in \( \mathbb{R}^2 \) if and only if \( X \) does not contain a copy of \( K_5, K_{3,3}, S^2 \), or the \( Q \)-space.

**Proof.** See [11], Theorem C'. □

**Theorem 12.** \( \clubsuit_{\sigma CP}(\mathbb{R}^2) = 6 \). That is, any \( \sigma \)-compact polyhedron which is 6-cleavable over \( \mathbb{R}^2 \) embeds in \( \mathbb{R}^2 \).

**Proof.** \( K_{3,3} \) is a compact CW-complex that is 5-cleavable over \( \mathbb{R}^2 \) but that cannot be embedded in \( \mathbb{R}^2 \); thus \( \clubsuit_{\sigma CP}(\mathbb{R}^2) \geq 6 \). If \( X \) is 6-cleavable over \( \mathbb{R}^2 \) then, by Lemma 3, \( X \) does not contain a copy of \( K_5, K_{3,3}, S^2 \), or the \( Q \)-space, so \( X \) embeds in \( \mathbb{R}^2 \) by Lemma 11. Thus \( \clubsuit_{\sigma CP}(\mathbb{R}^2) \leq 6 \). □

**Question 13.** Can we compute the cleavage of \( \mathbb{R}^2 \) with respect to a broader class of spaces? Specifically, what is the cleavage of \( \mathbb{R}^2 \) with respect to compact spaces (\( \sigma \)-compact spaces, continua, etc.)?

**Question 14.** Lemma 11 is a Kuratowski-like theorem for \( \sigma CP \) and \( \mathbb{R}^2 \): it gives a finite list of members of \( \sigma CP \) such that any \( X \in \sigma CP \) embeds in \( \mathbb{R}^2 \) if and only \( X \) does not contain as a subspace anything on this list. Is there a Kuratowski-like theorem for compact spaces and \( \mathbb{R}^2 \)? In other words, is there a finite list of compact spaces such that any compact space which does not embed in \( \mathbb{R}^2 \) contains something on this list as a subspace?

A negative answer to Question 14 could imply that the cleavage of \( \mathbb{R}^2 \) with respect to the compact spaces is infinite, so this question is related to Question 5. A negative answer would also give contrast to the currently open problem of whether it is consistent with ZFC that every compact Hausdorff space contains a copy of either \( \omega + 1 \) or \( \beta \mathbb{N} \). See [6], pp. 113 for more on this problem.

In the same way that Theorem 12 translates a Kuratowski-like theorem into a theorem about cleavage, we now prove a theorem that translates into a theorem about cleavage a celebrated minimality result from graph theory. For the statement of this theorem, let \( CG \) denote the class of compact graphs, which is the same as the class of graphs with finitely many vertices and edges.

**Theorem 15.** If \( Y \) is a surface (a 2-dimensional manifold) then \( \clubsuit_{CG}(Y) \) is finite. That is, for every compact 2-manifold \( Y \) there is a finite number \( n_Y \) such that whenever a compact graph \( G \) is \( n_Y \)-cleavable over \( Y \) then \( X \) embeds into \( Y \). Moreover, if \( Y \) is a surface of genus \( \gamma \) then

\[
 n_Y > \left\lfloor \frac{7 + \sqrt{48 \gamma + 1}}{2} \right\rfloor
\]
(Here \([x] \) denotes the greatest integer \(m\) such that \(m \leq x\).) Thus arbitrarily high finite numbers are represented as the cleavage of 2-manifolds with respect to \(CG\).

Proof. If \(Y\) is a surface then, by a result of Robertson and Seymour (see [12]), there is a finite list of graphs \(G_0, \ldots, G_n\) such that an arbitrary graph \(G\) embeds in \(Y\) if and only if it does not contain a homeomorphic copy of one of \(G_0, \ldots, G_n\). Moreover, for \(i = 0, \ldots, n\), \(\text{♣}(G_i, Y)\) is finite. To show this, it is enough to note that there is a finite partition of \(G_i\) which is not respected by any continuous function. One such partition is the partition which takes \(\{v\}\) as an equivalence class for each vertex \(v\) of \(G\) and which also takes each edge \(E\) of \(G\) (without the endpoints) to be its own equivalence class; the proof that no continuous function can respect this partition is analogous to (one direction of) the proof of Lemma 6 together with the fact that \(G_i\) does not embed in \(Y\). Evidently, \(\text{♣}_{CG}(Y) \leq \max_{i \leq n} \{\text{♣}(G_i, Y)\}\).

To prove the stated bound for a surface \(Y\) of genus \(\gamma\), it is enough to prove that there is some compact graph that is \(m\)-cleavable over \(Y\) but does not embed in \(Y\) (where \(m = \left[\frac{1}{2}(7 + \sqrt{48\gamma + 1})\right]\)). This formula, taken from [5], gives the greatest \(m\) such that \(K_m\) embeds in \(Y\); thus, by the same argument as in the proof of Theorem 7, every graph is \(m\)-cleavable over \(Y\). As not every graph embeds in \(Y\), this completes the proof. \(\square\)

Note that this lower bound for \(n_Y\) is not very strong because it only takes into account a single one of the excluded graphs for a surface of genus \(\gamma\). The number of excluded graphs for a given surface may be quite large, despite the fact that there are only two excluded graphs for \(\mathbb{R}^2\) (for example, the projective plane has 103 distinct excluded graphs). Even for the case of \(\mathbb{R}^2\) the inequality given above is not optimal.

**Question 16.** Can the exact value of \(\text{♣}_{CG}(Y)\) be computed for an arbitrary surface \(Y\)? What about the cleavage of surfaces with respect to a broader class of spaces, e.g., the compact spaces?

4. **The Spectrum of \(\mathbb{R}^2\)**

One general question we can ask about cleavage and cleavability is

**Question 17.** For a fixed space \(Y\), what are the possible values for \(\text{♣}(X, Y)\)? What if \(X\) is restricted to be in some fixed class \(C\)?

We call these sets of possible values the **spectrum** of \(Y\) and the spectrum of \(Y\) relative to \(C\), respectively. For example, Arhangel’skii proved (see [2], Proposition 3) that the spectrum of \(\mathbb{R}^\omega\) does not contain
any values $\lambda$ such that $2 < \lambda \leq c$. Together with Proposition 1(ii) and the fact that the discrete space on $c^+$ points is $c$-cleavable over $\mathbb{R}^\omega$, this shows that the spectrum of $\mathbb{R}^\omega$ is \{2, $c^+$, $\infty$\}. In fact, as Arhangel’skiĭ himself points out, his proof generalizes to show that the spectrum of $\mathbb{R}^\kappa$ is \{2, $(2^\kappa)^+$, $\infty$\} for any infinite $\kappa$. We have already seen that this result does not extend to the case $\kappa = 2$. Theorem 7 tells us that the spectrum of $\mathbb{R}^2$ with respect to countable graphs is \{5, 6, $\infty$\}. We have also seen (Lemma 10) that there are compact polyhedra that are not cleavable over $\mathbb{R}^2$, i.e., spaces $X$ such that ♦$(X, \mathbb{R}^2) = 2$, thus showing that the spectrum of $\mathbb{R}^2$ contains \{2, 5, 6, $\infty$\}. In this section we give two compact polyhedra, one that is 3-cleavable but not 4-cleavable over $\mathbb{R}^2$, and another that is cleavable but not 3-cleavable over $\mathbb{R}^2$. This will prove that the spectrum of $\mathbb{R}^2$ relative to $\sigma CP$ is \{2, 3, 4, 5, 6, $\infty$\}, and that the spectrum of $\mathbb{R}^2$ contains \{2, 3, 4, 5, 6, $\infty$\}.

Before moving on, we note the contrast between Arhangel’skiĭ’s result for $\mathbb{R}^\kappa$ and our result for $\mathbb{R}^2$. This contrast leads us naturally to the question of the intermediate cases:

**Question 18.** For $n \geq 3$, what is the spectrum of $\mathbb{R}^n$? With respect to compact spaces, compact polyhedra, or $\sigma$-compact polyhedra? In particular, does the spectrum of $\mathbb{R}^n$ contain values other than 2, $c^+$, and $\infty$ for all finite $n \geq 3$?

Define $D_{3,3}$ to be the topological space obtained by expanding each vertex of $K_{3,3}$ to a copy of the closed unit disc; the edges emanate from the boundary of the discs, so that the resulting space is locally embeddable in $\mathbb{R}^2$. Two pictures of this space are shown below. $D_{3,3}$ is a compact 2-dimensional polyhedron.
Let $f : X \to Y$ be any function. Define

$$M_f = \{x \in X : \exists y \in X \setminus \{x\} \text{ such that } f(x) = f(y)\}$$

That is, $M_f$ is the set of points on which $f$ is not injective. We will say that $f : X \to Y$ is **almost injective** if $|M_f| < |X|$.

**Lemma 19.** Let $U$ denote the open unit disc. There exist $A, B, C \subseteq U$ such that, if $f : U \to \mathbb{R}^2$ is any continuous function satisfying

$$f[A] \cap f[B] = f[A] \cap f[C] = f[B] \cap f[C] = \emptyset$$

then $f$ is almost injective. Moreover, we may take $A, B,$ and $C$ such that all of the following hold:

- $A \cap B = A \cap C = B \cap C = \emptyset$
- $A \cup B \cup C = U$
- none of $A, B,$ or $C$ contains an arc
- if $h_0, h_1$ are embeddings $U \to \mathbb{R}^2$ with $h_0 \neq h_1$, then
  $$h_0[U] \cap h_1[U] = \emptyset \iff h_0[A] \cap h_1[B] = \emptyset \iff h_0[A] \cap h_1[C] = \emptyset \iff h_0[B] \cap h_1[C] = \emptyset$$

**Proof.** We build the sets $A$, $B$, and $C$ simultaneously by transfinite recursion. Let $\langle f_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of all continuous functions $f : U \to \mathbb{R}^2$ that are not almost injective, i.e., such that

$$|\{x \in U : \exists y \in U \setminus \{x\} \text{ such that } f(x) = f(y)\}| = \mathfrak{c}$$

(Recall that every continuous function $U \to \mathbb{R}^2$ is determined by its action on a dense subset; it follows that there are only $\mathfrak{c}$-many such functions.) Let $\langle g_\alpha : \alpha < \mathfrak{c} \rangle$ be an enumeration of all embeddings $[0, 1] \to U$. Lastly, let $\langle (h_0^\alpha, h_1^\alpha) : \alpha < \mathfrak{c} \rangle$ be an enumeration of all pairs $(h_0, h_1)$ of embeddings $U \to \mathbb{R}^2$ such that $h_0[U] \cap h_1[U] \neq \emptyset$. Since every subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}^2$ is open, $h_0^\alpha[U] \cap h_1^\alpha[U]$ is open, and hence of cardinality $\mathfrak{c}$, for every $\alpha < \mathfrak{c}$.

As the base case for our recursion, take $A^0 = B^0 = C^0 = \emptyset$.

Let $\alpha < \mathfrak{c}$ and suppose that we have constructed three sets $A^\alpha$, $B^\alpha$, and $C^\alpha$ which are pairwise disjoint and of cardinality strictly less than $\mathfrak{c}$. First, let $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, and $z_2$ be any distinct points of $U \setminus (A^\alpha \cup B^\alpha \cup C^\alpha)$ such that

$$f_\alpha(x_1) = f_\alpha(y_1) \quad f_\alpha(x_2) = f_\alpha(z_1) \quad f_\alpha(y_2) = f_\alpha(z_2)$$

Some such points must exist since $f_\alpha$ is not almost injective. Next, let $x_3$, $y_3$, and $z_3$ be any three distinct points of $g_\alpha[0, 1] \setminus (A^\alpha \cup B^\alpha \cup C^\alpha \cup \{x_1, x_2, y_1, y_2, z_1, z_2\})$. Lastly, set $V = h_0^\alpha[U] \cap h_1^\alpha[U]$ and let

$$x_4, y_4, z_4 \in (h_0^\alpha)^{-1}[V] \setminus (A^\alpha \cup B^\alpha \cup C^\alpha)$$
such that all these points are distinct, both from each other and from
$x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$, and such that

\[ h_\alpha^0(x_4) = h_\alpha^1(y_4) \quad h_\alpha^0(x_5) = h_\alpha^1(z_4) \quad h_\alpha^0(y_5) = h_\alpha^1(z_5) \]

Some such points must exist because \(|V| = c\). Let

\[
A^{\alpha+1} = A^\alpha \cup \{x_1, x_2, x_3, x_4\} \quad B^{\alpha+1} = B^\alpha \cup \{y_1, y_2, y_3, y_4\} \\
C^{\alpha+1} = C^\alpha \cup \{z_1, z_2, z_3, z_4\}
\]

This defines our construction at successor stages.

If \(\alpha\) is a limit ordinal, take \(A^\alpha = \bigcup_{\beta<\alpha} A^\beta\), \(B^\alpha = \bigcup_{\beta<\alpha} B^\beta\), and \(C^\alpha = \bigcup_{\beta<\alpha} C^\beta\).

Taking \(A = \bigcup_{\alpha<\xi} A^\alpha\), \(B = \bigcup_{\alpha<\xi} B^\alpha\), and \(C = U \setminus (A \cup B)\), it is clear from our construction that \(A\), \(B\), and \(C\) satisfy the desired conclusions. \(\square\)

**Lemma 20** (Arhangel'skiï). Let \(n \in \mathbb{N}\) and let \(f : \mathbb{R}^n \to \mathbb{R}^n\) be an almost injective continuous mapping. Then \(f\) is an embedding.

**Proof.** See [2], Lemma 3. \(\square\)

**Theorem 21.** \(\clubsuit(D_{3,3}) = 3\).

**Proof.** In what follows, we will denote the six (maximal) closed discs in \(D_{3,3}\) by \(D_1, ..., D_6\), and we will them such that \(D_i\) and \(D_j\) are connected by an edge if and only if \(i \leq 3\) and \(j > 3\) or \(j \leq 3\) and \(i > 3\).

We will use the term “edge” in the obvious sense: an edge is a copy of \([0,1]\) inside \(D_{3,3}\), where one of the endpoints is identified with the boundary of one of the six closed discs, the other endpoint is identified with the boundary of a different one (thus \(|D_i \cap E_{i,j}| = 1\)), and, except for the endpoints, no point of any edge is contained in any of the \(D_i\).

The edge connecting \(D_i\) and \(D_j\) will be denoted \(E_{i,j}\).

First we show that \(D_{3,3}\) is cleavable over \(\mathbb{R}^2\). Let \(\mathcal{P}\) be any 2-partition of \(D_{3,3}\); then there must be two non-adjacent edges, \(E\) and \(E'\), which contain points \(p \in E, p' \in E'\), other than their endpoints, in the same equivalence class. It is easy to see that we can then find a continuous function \(f : D_{3,3} \to \mathbb{R}^2\) which is injective at every point except that \(f(p) = f(p')\) (one such near-embedding is given in the rightmost of the two pictures of \(D_{3,3}\) shown above); \(f\) is a continuous function which respects \(\mathcal{P}\). Thus \(\clubsuit(D_{3,3}, \mathbb{R}^2) \geq 3\).

Next we show that \(D_{3,3}\) is not 3-cleavable over \(\mathbb{R}^2\). To do this, it will be easier to show that a proper subset of \(D_{3,3}\) is not 3-cleavable over
$\mathbb{R}^2$, which is sufficient by Lemma 3. Let

$$U_{3,3} = \bigcup_{i=1}^{6} (D_i \setminus \partial D_i) \cup \bigcup_{i=1}^{3} \bigcup_{j=4}^{6} E_{i,j}$$

In other words, $U_{3,3}$ looks just like $D_{3,3}$ except that we have used six open discs, rather than six closed discs, in its construction. In what follows, we will refer to $D_i \setminus \partial D_i$ as $U_i$.

We construct a 3-partition of $U_{3,3}$ which is not respected by any continuous function into $\mathbb{R}^2$. For each $U_i$, let $A_i$, $B_i$, and $C_i$ denote the three subsets of $U_i$ described in Lemma 19. Partition $U_{3,3}$ as follows:

$$E_{1,4} \cup E_{1,5} \cup E_{1,6} \cup A_1 \cup A_2 \cup B_3 \cup B_4 \cup C_5 \cup C_6$$

$$E_{2,4} \cup E_{2,5} \cup E_{2,6} \cup A_3 \cup A_5 \cup B_1 \cup B_6 \cup C_2 \cup C_4$$

$$E_{3,4} \cup E_{3,5} \cup E_{3,6} \cup A_4 \cup A_6 \cup B_2 \cup B_5 \cup C_1 \cup C_3$$

Suppose that $f : D_{3,3} \to \mathbb{R}^2$ is a continuous function which respects this partition. By Lemma 19 and our choice of partition, $f \mid U_i$ is almost injective for each $i$, and it follows from Lemma 20 that $f \mid U_i$ is a homeomorphism. It then follows from the last assertion of Lemma 19 and our choice of partition that, if $i \neq j$, $f[U_i] \cap f[U_j] = \emptyset$.

Following the argument of the proof of Lemma 6, we may assume that $f$ maps each $E_{i,j}$ either to a homeomorphic copy of $[0, 1]$ or to a single point. Now, reasoning as in the easy direction of Kuratowski’s Theorem, we see that either two non-adjacent edges must cross or some edge must pass through one of the $U_i$. No two non-adjacent edges may cross since no two non-adjacent edges are in the same equivalence class of our partition. No edge may pass through any of the $U_i$ since none of the $A$, $B$, or $C$ contains an arc and since we have constructed our partition in such a way that, for any fixed $i$, a given edge $E$ is only in the same partition as $A_i$, $B_i$, or $C_i$, but never two of these. Thus it is impossible for $f$ to respect our partition, and we conclude that $\clubsuit(D_{3,3}, \mathbb{R}^2) \leq \clubsuit(U_{3,3}, \mathbb{R}^2) \leq 3$.

\[ \square \]

**Theorem 22.** There is a compact 2-dimensional polyhedron $X$ such that $\clubsuit(X, \mathbb{R}^2) = 4$.

**Proof.** We construct $X$ in a manner analogous to the construction of $D_{3,3}$, except that only two vertices of $K_{3,3}$, rather than all six, are expanded to discs. Thus $X$ is composed of two discs, $D_1$ and $D_2$, four points, $v_1$, $v_2$, $v_3$, and $v_4$, and nine edges. (Nb: This description does not define $X$ uniquely up to homeomorphism, since we could have either an edge connecting $D_1$ and $D_2$ or not. It does not matter for our purposes which of these two spaces $X$ is chosen to be.)
If $P$ is a 3-partition of $X$ then at least two of $v_1$, $v_2$, $v_3$, and $v_4$ are in the same equivalence class. It is then easy to show, as in the proof of Lemma 6, that there is a continuous function $f : X \to \mathbb{R}^2$ which identifies these two vertices but is otherwise injective. Thus $X$ is 3-cleavable over $\mathbb{R}^2$, i.e., $\bullet(X, \mathbb{R}^2) \geq 4$.

The proof that $\bullet(X, \mathbb{R}^2) \leq 4$ is essentially the same as the proof that $\bullet(D_{3,3}, \mathbb{R}^2) \leq 3$ (Theorem 21) and we omit the details. \hfill \Box

**Corollary 23.** The spectrum of $\mathbb{R}^2$ with respect to compact polyhedra and $\sigma$-compact polyhedra is $\{2, 3, 4, 5, 6, \infty\}$.

**Question 24.** Is there a compact space $X$ which does not embed in $\mathbb{R}^2$ but satisfies $\bullet(X, \mathbb{R}^2) > 6$?

**Question 25.** Under what circumstances is it the case that, if $\kappa$ is in the spectrum of $Y$ relative to compact polyhedra (or $\sigma$-compact polyhedra, or compact spaces generally), then so is any $\lambda < \kappa$?

**References**