Ramsey-type results for dynamical systems

Will Brian

11th AIMS Conference on Dynamical Systems, Differential Equations, and Applications

Orlando
3 July 2016
Can every sequence be orbit-like?

A *dynamical system* is a compact metric space $X$ together with a continuous map $f : X \to X$. 
Can every sequence be orbit-like?

A dynamical system is a compact metric space $X$ together with a continuous map $f : X \rightarrow X$.

Question: In which dynamical systems does every sequence of points look like an orbit?
Can every sequence be orbit-like?

A *dynamical system* is a compact metric space $X$ together with a continuous map $f : X \to X$.

**Question:** *In which dynamical systems does every sequence of points look like an orbit?*

- What does it mean for a given sequence of points in $X$ to “look like” an orbit?
A dynamical system is a compact metric space $X$ together with a continuous map $f : X \to X$.

**Question:** In which dynamical systems does every sequence of points look like an orbit?

- What does it mean for a given sequence of points in $X$ to “look like” an orbit?
- A little more generally, what does it mean for two sequences to “look like” each other?
Can every sequence be orbit-like?

A *dynamical system* is a compact metric space $X$ together with a continuous map $f : X \to X$.

**Question:** In which dynamical systems does every sequence of points look like an orbit?

- What does it mean for a given sequence of points in $X$ to “look like” an orbit?
- A little more generally, what does it mean for two sequences to “look like” each other?

*Roughly, two sequences look alike if they are close frequently.*
Shadowing and asymptotic shadowing

Let $\langle x_n : n \in \mathbb{N} \rangle$ and $\langle y_n : n \in \mathbb{N} \rangle$ be sequences of points in a compact metric space $X$ (with metric $d$), and let $A \subseteq \mathbb{N}$. 
Shadowing and asymptotic shadowing

Let \( \langle x_n : n \in \mathbb{N} \rangle \) and \( \langle y_n : n \in \mathbb{N} \rangle \) be sequences of points in a compact metric space \( X \) (with metric \( d \)), and let \( A \subseteq \mathbb{N} \).

- Given \( \varepsilon > 0 \), we say that these sequences \( \varepsilon \)-shadow each other on \( A \) if
  \[
  \{ n \in \mathbb{N} : d(x_n, y_n) < \varepsilon \} \supseteq A.
  \]
Shadowing and asymptotic shadowing

Let $\langle x_n : n \in \mathbb{N} \rangle$ and $\langle y_n : n \in \mathbb{N} \rangle$ be sequences of points in a compact metric space $X$ (with metric $d$), and let $A \subseteq \mathbb{N}$.

- Given $\varepsilon > 0$, we say that these sequences $\varepsilon$-shadow each other on $A$ if
  \[ \{ n \in \mathbb{N} : d(x_n, y_n) < \varepsilon \} \supseteq A. \]

- We say that these sequences are asymptotic on $A$ if
  \[ \lim_{n \in A} d(x_n, y_n) = 0, \]
  or, equivalently, if for every $\varepsilon > 0$ the sequences $\varepsilon$-shadow each other on a co-finite subset of $A$. 

A question
Defining “close”
Defining “frequently”
$2 \times 2 = 4$ answers
Filters and families

A filter on \( \mathbb{N} \) is a collection \( \mathcal{F} \) of subsets of \( \mathbb{N} \) such that

- (nontriviality) \( \emptyset \notin \mathcal{F} \), \( \mathbb{N} \in \mathcal{F} \).
- (upwards heredity) if \( A \in \mathcal{F} \) and \( A \subseteq B \), then \( B \in \mathcal{F} \).
- (finite intersection property) if \( A, B \in \mathcal{F} \), then \( A \cap B \in \mathcal{F} \).
Filters and families

A filter on \( \mathbb{N} \) is a collection \( \mathcal{F} \) of subsets of \( \mathbb{N} \) such that

- (nontriviality) \( \emptyset \notin \mathcal{F}, \mathbb{N} \in \mathcal{F} \).
- (upwards heredity) if \( A \in \mathcal{F} \) and \( A \subseteq B \), then \( B \in \mathcal{F} \).
- (finite intersection property) if \( A, B \in \mathcal{F} \), then \( A \cap B \in \mathcal{F} \).

A Furstenberg family (or semifilter) is a collection \( \mathcal{F} \) satisfying only the first two of these axioms.
Filters and families

A **filter** on \( \mathbb{N} \) is a collection \( \mathcal{F} \) of subsets of \( \mathbb{N} \) such that

- (nontriviality) \( \emptyset \notin \mathcal{F}, \mathbb{N} \in \mathcal{F} \).
- (upwards heredity) if \( A \in \mathcal{F} \) and \( A \subseteq B \), then \( B \in \mathcal{F} \).
- (finite intersection property) if \( A, B \in \mathcal{F} \), then \( A \cap B \in \mathcal{F} \).

A **Furstenberg family** (or **semifilter**) is a collection \( \mathcal{F} \) satisfying only the first two of these axioms.

Every family \( \mathcal{F} \) has a **dual**

\[
\mathcal{F}^* = \{ A \subseteq \mathbb{N} : \text{for every } B \in \mathcal{F}, A \cap B \neq \emptyset \}.
\]
Partition regularity

A family $\mathcal{F}$ is called \emph{partition regular} if for every $A \in \mathcal{F}$, if $A = \bigcup_{i \leq n} A_i$, then $A_i \in \mathcal{F}$ for some $i \leq n$. 
Partition regularity

A family $F$ is called \textit{partition regular} if for every $A \in F$, if $A = \bigcup_{i \leq n} A_i$, then $A_i \in F$ for some $i \leq n$.

For example, the family of infinite sets, the family of sets with nonzero upper density, the family of sets containing arbitrarily long arithmetic sequences, the family of IP sets, and the family of piecewise syndetic sets are all partition regular.
Partition regularity

A family $\mathcal{F}$ is called \textit{partition regular} if for every $A \in \mathcal{F}$, if $A = \bigcup_{i \leq n} A_i$, then $A_i \in \mathcal{F}$ for some $i \leq n$.

For example, the family of infinite sets, the family of sets with nonzero upper density, the family of sets containing arbitrarily long arithmetic sequences, the family of IP sets, and the family of piecewise syndetic sets are all partition regular.

\textbf{Theorem (Glasner)}

1. A family $\mathcal{F}$ is partition regular if and only if it is dual to a filter.
2. If $\mathcal{F}$ is a partition regular family, then there is ultrafilter $p \subseteq \mathcal{F}$.
We are now ready to answer the question: What does it mean for two sequences to be frequently close?
A working answer

We are now ready to answer the question: *What does it mean for two sequences to be frequently close?*

- Given a partition regular family $\mathcal{F}$ (i.e., a coherent notion of “frequently”), we may insist that for all $\varepsilon > 0$ the sequences $\varepsilon$-shadow each other on a set in $\mathcal{F}$. 
We are now ready to answer the question: *What does it mean for two sequences to be frequently close?*

- Given a partition regular family $\mathcal{F}$ (i.e., a coherent notion of “frequently”), we may insist that for all $\varepsilon > 0$ the sequences $\varepsilon$-shadow each other on a set in $\mathcal{F}$.
- Alternatively, we may insist that the sequences are asymptotic on a set in $\mathcal{F}$. 
We are now ready to answer the question: \textit{What does it mean for two sequences to be frequently close?}

- Given a partition regular family $\mathcal{F}$ (i.e., a coherent notion of “frequently”), we may insist that for all $\varepsilon > 0$ the sequences $\varepsilon$-shadow each other on a set in $\mathcal{F}$.

- Alternatively, we may insist that the sequences are asymptotic on a set in $\mathcal{F}$.

This leads us to two definitions and two definition-schema:
Four properties

- \((X, f)\) has **Ramsey shadowing** if for every sequence \(\xi\) of points in \(X\), every partition regular family \(\mathcal{F}\), and every \(\varepsilon > 0\), there is some \(x \in X\) such that \(\xi\) and \(\langle f^n(x) : n \in \mathbb{N} \rangle\) \(\varepsilon\)-shadow each other on a set in \(\mathcal{F}\).
Four properties

- \((X, f)\) has **Ramsey shadowing** if for every sequence \(\xi\) of points in \(X\), every partition regular family \(\mathcal{F}\), and every \(\varepsilon > 0\), there is some \(x \in X\) such that \(\xi\) and \(\langle f^n(x) : n \in \mathbb{N} \rangle\) \(\varepsilon\)-shadow each other on a set in \(\mathcal{F}\).

- \((X, f)\) has **asymptotic Ramsey shadowing** if for every sequence \(\xi\) of points in \(X\) and every partition regular family \(\mathcal{F}\), there is some \(x \in X\) such that \(\xi\) and \(\langle f^n(x) : n \in \mathbb{N} \rangle\) are asymptotic on a set in \(\mathcal{F}\).
Comparing arbitrary sequences and orbits (asymptotic) Ramsey shadowing
Those other two properties: partial results

Four properties

- $(X, f)$ has *Ramsey shadowing* if for every sequence $\xi$ of points in $X$, every partition regular family $\mathcal{F}$, and every $\varepsilon > 0$, there is some $x \in X$ such that $\xi$ and $\langle f^n(x) : n \in \mathbb{N} \rangle$ $\varepsilon$-shadow each other on a set in $\mathcal{F}$.

- $(X, f)$ has *asymptotic Ramsey shadowing* if for every sequence $\xi$ of points in $X$ and every partition regular family $\mathcal{F}$, there is some $x \in X$ such that $\xi$ and $\langle f^n(x) : n \in \mathbb{N} \rangle$ are asymptotic on a set in $\mathcal{F}$.

- Alternatively, we may fix $\mathcal{F}$ beforehand. This provides us with the notions of $\mathcal{F}$-Ramsey shadowing and asymptotic $\mathcal{F}$-Ramsey shadowing.
Comparing arbitrary sequences and orbits
( asymptotic) Ramsey shadowing
Those other two properties: partial results

Four properties

- \((X, f)\) has \textit{Ramsey shadowing} if for every sequence \(\xi\) of points in \(X\), every partition regular family \(\mathcal{F}\), and every \(\varepsilon > 0\), there is some \(x \in X\) such that \(\xi\) and \(\langle f^n(x) : n \in \mathbb{N}\rangle\) \(\varepsilon\)-shadow each other on a set in \(\mathcal{F}\).

- \((X, f)\) has \textit{asymptotic Ramsey shadowing} if for every sequence \(\xi\) of points in \(X\) and every partition regular family \(\mathcal{F}\), there is some \(x \in X\) such that \(\xi\) and \(\langle f^n(x) : n \in \mathbb{N}\rangle\) are asymptotic on a set in \(\mathcal{F}\).

- Alternatively, we may fix \(\mathcal{F}\) beforehand. This provides us with the notions of \(\mathcal{F}\)-Ramsey shadowing and asymptotic \(\mathcal{F}\)-Ramsey shadowing. For simplicity, in what follows we will only consider these notions for the family \(\mathcal{F}\) of infinite sets (but most of the results generalize).
Ramsey shadowing is equivalent to the following seemingly weaker alternative version: for every sequence $\xi$ of points in $X$, every ultrafilter $p$, and every $\varepsilon > 0$, there is some $x \in X$ such that $\xi$ and $\langle f^n(x) : n \in \mathbb{N} \rangle$ $\varepsilon$-shadow each other on a set in $p$. The analogous result also holds for asymptotic Ramsey shadowing.
Comparing arbitrary sequences and orbits
(asymptotic) Ramsey shadowing
Those other two properties: partial results

From partition regular families to ultrafilters

Lemma

Ramsey shadowing is equivalent to the following seemingly weaker alternative version: for every sequence $\xi$ of points in $X$, every ultrafilter $p$, and every $\varepsilon > 0$, there is some $x \in X$ such that $\xi$ and $\langle f^n(x) : n \in \mathbb{N} \rangle$ $\varepsilon$-shadow each other on a set in $p$.

The analogous result also holds for asymptotic Ramsey shadowing.

Proof.

For the forward implication, simply observe that every ultrafilter is partition regular. For the converse implication, use part (2) of the aforementioned result of Glasner.
Ramsey shadowing

Theorem (B. & Oprocha)

A dynamical system has Ramsey shadowing if and only if it has a dense set of minimal points.
Ramsey shadowing

Theorem (B. & Oprocha)

A dynamical system has Ramsey shadowing if and only if it has a dense set of minimal points.

In fact, this theorem is true for non-metrizable dynamical systems as well (after the definition of Ramsey shadowing has been suitably generalized).
asymptotic Ramsey shadowing

Recall that a dynamical system is *distal* if no two orbits are asymptotic on an infinite set.
Recall that a dynamical system is *distal* if no two orbits are asymptotic on an infinite set.

**Theorem**

The following are equivalent in any dynamical system \((X, f)\):

1. asymptotic Ramsey shadowing.
2. distality.
3. every member of the enveloping semigroup of \((X, f)\) is surjective.
4. for every ultrafilter \(p\), the map \(x \mapsto p\lim_{n \in \omega} f^n(x)\) is surjective.
Given \((X, f)\), let us say that \(A \subseteq \mathbb{N}\) has dense (surjective) limits if
\[
\left\{ \lim_{k \to \infty} f^{a_k}(x) : a_k \in A \text{ for all } k, \text{ and this limit exists} \right\}
\]
is dense in \(X\) (all of \(X\)).

**Theorem**

A dynamical system has (infinite sets)-Ramsey shadowing if and only if for every finite coloring of \(\mathbb{N}\), there is a monochromatic \(A \subseteq \mathbb{N}\) with dense limits.
Theorem

Asymptotic (infinite sets)-Ramsey shadowing implies that for every finite coloring of $\mathbb{N}$, there is a monochromatic $A \subseteq \mathbb{N}$ with surjective limits.
Asymptotic (infinite sets)-Ramsey shadowing implies that for every finite coloring of \( \mathbb{N} \), there is a monochromatic \( A \subseteq \mathbb{N} \) with surjective limits. It is implied by
Asymptotic (infinite sets)-Ramsey shadowing implies that for every finite coloring of \( \mathbb{N} \), there is a monochromatic \( A \subseteq \mathbb{N} \) with surjective limits. It is implied by

- the enveloping semigroup of \((X, f)\) contains a non-isolated surjective function (true for any weakly rigid system)
asymptotic (infinite sets)-Ramsey shadowing

**Theorem**

Asymptotic (infinite sets)-Ramsey shadowing implies that for every finite coloring of \( \mathbb{N} \), there is a monochromatic \( A \subseteq \mathbb{N} \) with surjective limits. It is implied by

- the enveloping semigroup of \((X, f)\) contains a non-isolated surjective function (true for any weakly rigid system)
- asymptotic shadowing plus chain recurrence (true for (nontrivial) shifts of finite type, dendritic Julia sets, and certain circular Julia sets)
An *asymptotic pseudo-orbit* (or *a.p.o.*) in $(X, f)$ is a sequence $\langle x_n : n \in \mathbb{N} \rangle$ of points in $X$ with the property that

$$\lim_{n \to \infty} d(f(x_n), x_{n+1}) = 0.$$
Asymptotic shadowing

An *asymptotic pseudo-orbit* (or a.p.o.) in \((X, f)\) is a sequence \(\langle x_n : n \in \mathbb{N} \rangle\) of points in \(X\) with the property that

\[
\lim_{n \to \infty} d(f(x_n), x_{n+1}) = 0.
\]

\((X, f)\) has the *asymptotic shadowing property* if every a.p.o. is asymptotic to an orbit on \(\mathbb{N}\): i.e., there is some \(x \in X\) such that

\[
\lim_{n \to \infty} d(f^n(x), x_n) = 0.
\]
An $\varepsilon$-chain from $a$ to $b$ is a finite sequence $\langle x_i : i \leq n \rangle$, with $n \geq 2$, such that $x_0 = a$, $x_n = b$, and

$$d(f(x_i), x_{i+1}) < \varepsilon$$

for all $i < n$. 

Chain recurrence
An \( \varepsilon \)-chain from \( a \) to \( b \) is a finite sequence \( \langle x_i : i \leq n \rangle \), with \( n \geq 2 \), such that \( x_0 = a \), \( x_n = b \), and

\[
d(f(x_i), x_{i+1}) < \varepsilon
\]

for all \( i < n \).

\((X, f)\) is chain recurrent if for every \( \varepsilon > 0 \) and every \( x \in X \), there is an \( \varepsilon \)-chain from \( x \) to itself.
Proof sketch

**Theorem**

If \((X, f)\) has asymptotic shadowing and chain recurrence, then it also has asymptotic (infinite sets)-Ramsey shadowing.

**Proof sketch.**

Let \(\langle x_n : n \in \mathbb{N} \rangle\) be an arbitrary sequence of points in \(X\). Let \(x\) be a limit point of \(\{x_n! : n \in \mathbb{N}\}\), and fix \(\langle n_k : k \in \mathbb{N} \rangle\) such that \(x_{n_k}! \to x\).
Theorem

If \((X, f)\) has asymptotic shadowing and chain recurrence, then it also has asymptotic (infinite sets)-Ramsey shadowing.

Proof sketch.

Let \(\langle x_n : n \in \mathbb{N} \rangle\) be an arbitrary sequence of points in \(X\). Let \(x\) be a limit point of \(\{x_n! : n \in \mathbb{N}\}\), and fix \(\langle n_k : k \in \mathbb{N} \rangle\) such that \(x_{n_k}! \to x\). For every \(m > 0\), let \(\chi_m\) be a \(\frac{1}{m}\)-chain from \(x\) to itself, but with the initial \(x\) deleted. Let \(\ell_m\) denote the length of \(\chi_m\).
Comparing arbitrary sequences and orbits (asymptotic) Ramsey shadowing
Those other two properties: partial results

Proof sketch

**Theorem**

*If* \((X, f)\) *has asymptotic shadowing and chain recurrence, then it also has asymptotic (infinite sets)-Ramsey shadowing.*

**Proof sketch.**

Let \(\langle x_n : n \in \mathbb{N} \rangle\) be an arbitrary sequence of points in \(X\). Let \(x\) be a limit point of \(\{x_n! : n \in \mathbb{N}\}\), and fix \(\langle n_k : k \in \mathbb{N} \rangle\) such that \(x_{n_k!} \to x\). For every \(m > 0\), let \(\chi_m\) be a \(\frac{1}{m}\)-chain from \(x\) to itself, but with the initial \(x\) deleted. Let \(\ell_m\) denote the length of \(\chi_m\).

By concatenating these chains, we can build an a.p.o. in \(X\):

\[
\left(\chi_1 \cdots \chi_1\right) \left(\chi_2 \cdots \chi_2\right) \left(\chi_3 \cdots \chi_3\right) \cdots
\]
Proof sketch (continued)

Proof sketch.

If $N_m$ denotes the number of copies of $\chi_m$ used, then we need two things:

- $N_1 + N_2 + \cdots + N_m = n_k!$ for some $k$, and
- $N_1 + N_2 + \cdots + N_m$ is a multiple of $\ell_{m+1}$. 

Will Brian

Ramsey-type results for dynamical systems
Proof sketch. If $N_m$ denotes the number of copies of $\chi_m$ used, then we need two things:

- $N_1 + N_2 + \cdots + N_m = n_k!$ for some $k$, and
- $N_1 + N_2 + \cdots + N_m$ is a multiple of $\ell_{m+1}$.

The second condition allows us to achieve the first condition again at the next stage of the construction. The first condition ensures that for infinitely many $k$, the $(n_k!)^{th}$ member of our a.p.o. is $x$. If $y$ is a point witnessing the asymptotic shadowing property for this a.p.o., then $\langle x_n : n \in \mathbb{N} \rangle$ and the orbit of $y$ are asymptotic on an infinite set.
Proof sketch.

If $N_m$ denotes the number of copies of $\chi_m$ used, then we need two things:

- $N_1 + N_2 + \cdots + N_m = n_k!$ for some $k$, and
- $N_1 + N_2 + \cdots + N_m$ is a multiple of $\ell_{m+1}$.

The second condition allows us to achieve the first condition again at the next stage of the construction. The first condition ensures that for infinitely many $k$, the $(n_k!)^{th}$ member of our a.p.o. is $x$. If $y$ is a point witnessing the asymptotic shadowing property for this a.p.o., then $\langle x_n : n \in \mathbb{N} \rangle$ and the orbit of $y$ are asymptotic on an infinite set.
Comparing arbitrary sequences and orbits
(asymptotic) Ramsey shadowing
Those other two properties: partial results

The end

Thanks for listening!