The Stone–Cech compactification
Dynamics and algebra in $\mathbb{N}^*$

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What is $\beta \mathbb{N}$?

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\end{array}
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- $\beta\mathbb{N}$ and $\mathbb{N}^*$ are compact Hausdorff spaces.
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- $\beta\mathbb{N}$ and $\mathbb{N}^*$ have cardinality $2^{\aleph_0}$. 
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We can think of the points of $\mathbb{N}^*$ as operators that map sequences to their limit points.

- Let $X$ be any compact Hausdorff space, and let $\langle x_n : n \in \mathbb{N} \rangle$ be a sequence of points in $X$.
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- Each $p \in \mathbb{N}^*$ corresponds to a unique limit point of this sequence, denoted $p\lim_{n \in \mathbb{N}} x_n$.
- The function $f(n) = x_n$ has a (unique) Stone extension $\beta f : \beta \mathbb{N} \to X$. $p\lim_{n \in \mathbb{N}} x_n$ is defined to be $\beta f(p)$. 
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- The function \( f(n) = x_n \) has a (unique) Stone extension \( \beta f : \beta \mathbb{N} \to X \). \( p\text{-lim}_{n \in \mathbb{N}} x_n \) is defined to be \( \beta f(p) \).
- Very roughly, these operators are telling us how convergence happens along a sequence.
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- Taking limits in $\beta\mathbb{N}$, we have $p = \text{p-lim}_{n \in \mathbb{N}} n$ and $\sigma(p) = \text{p-lim}_{n \in \mathbb{N}} (n + 1)$.
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- Taking limits in $\beta \mathbb{N}$, we have $p = p\text{-}\lim_{n \in \mathbb{N}} n$ and $\sigma(p) = p\text{-}\lim_{n \in \mathbb{N}} (n + 1)$.
- $\sigma$ is a homeomorphism from $\beta \mathbb{N}$ to itself, and it restricts to a homeomorphism of $\mathbb{N}^*$ to itself.
(X, f) is a *quotient* of (X, f) if there is a continuous surjection Q : X → Y such that Q ∘ f = g ∘ Q.
The universality of $\mathbb{N}^*$

- $(X, f)$ is a *quotient* of $(X, f)$ if there is a continuous surjection $Q : X \to Y$ such that $Q \circ f = g \circ Q$.
- If $(X, f)$ is a dynamical system and $x \in X$, then $\omega(x)$ is the set of limit points of the orbit of $x$, i.e.,

$$\omega(x) = \bigcap_{n \in \mathbb{N}} \{f^m(x) : m \geq n\}.$$
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**Theorem**

*If $(X, f)$ is any dynamical system and $x \in X$, then $(\omega(x), f)$ is a quotient of $(\mathbb{N}^*, \sigma)$.***
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Theorem

If $(X, f)$ is any dynamical system and $x \in X$, then $(\omega(x), f)$ is a quotient of $(\mathbb{N}^*, \sigma)$. Conversely, every quotient of $(\mathbb{N}^*, \sigma)$ is isomorphic to the $\omega$-limit set of some point in some dynamical system.
For $p, q \in \mathbb{N}^*$, define $p + q = \lim_{m \in \mathbb{N}} \sigma^n(p)$. 

\[ \text{This operation makes} \quad \mathbb{N}^* \quad \text{into a left-topological semigroup.} \]

\[ \text{The semigroup structure of} \quad \mathbb{N}^* \quad \text{enjoys a certain universal status in semigroup theory, analogous to the status of} \quad \left( \mathbb{N}^*, \sigma \right) \quad \text{among dynamical systems.} \]

\[ \text{This semigroup structure can be exploited to prove results in} \quad \text{combinatorics and Ramsey theory:} \]

- van der Waerden's Theorem
- Hales-Jewett Theorem
- Hindman's Theorem
- Central sets theorem
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For \( p, q \in \mathbb{N}^\ast \), define
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This operation makes \( \mathbb{N}^\ast \) into a left-topological semigroup.

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Strange behavior

**Theorem**

*It is consistent with ZFC that some minimal right ideal of \( \mathbb{N}^* \) is a \( P \)-set.*
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**Corollary**

*It is consistent with ZFC that*

1. \( \mathbb{N}^* \) *has minimal right ideals that are also prime.*
2. \( \mathbb{N}^* \) *has minimal idempotents that are also maximal.*
3. *the minimal right ideals of \( \mathbb{N}^* \) are not homeomorphically embedded (though they are homeomorphic).*
An $\varepsilon$-pseudo-orbit (or $\varepsilon$-chain) is an orbit computed with an error of $\varepsilon$ at each step.
Chain transitivity with a metric

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A metrizable dynamical system is chain transitive if for any $x, y \in X$ and any $\varepsilon > 0$, there is an $\varepsilon$-chain from $x$ to $y$. 

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Dynamics in $\mathbb{N}^*$
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Given an open cover $\mathcal{U}$ of $X$, a $\mathcal{U}$-chain is an orbit computed with “errors” in $\mathcal{U}$; i.e., it is a sequence $\langle x_i : i \leq n \rangle$ such that for every $i < n$ there is some $U \in \mathcal{U}$ with $f(x_i), x_{i+1} \in U$. 

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- $X$ is chain transitive if for every open cover $\mathcal{U}$ of $X$ and every $x, y \in X$, there is a $\mathcal{U}$-chain connecting $x$ and $y$. 

**Theorem**

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We can also make sense of chains and chain transitivity in non-metrizable systems:

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**Theorem**

$\mathbb{N}^*$ is chain transitive.
Chain transitivity matters

Theorem

\textit{It is consistent with and independent of ZFC that the shift map and its inverse are (up to isomorphism) the only chain transitive autohomeomorphisms} \( \mathbb{N}^* \).

Theorem

\textit{If} \( p > \kappa \) \textit{and if} \( X \) \textit{has weight at most} \( \kappa \), \textit{then} \((X, f)\) \textit{is a quotient of} \((\mathbb{N}^*, \sigma)\) \textit{if and only if} \((X, f)\) \textit{is chain transitive}.

Corollary

\textit{If} \( X \) \textit{is metrizable, then} \((X, f)\) \textit{is a quotient of} \((\mathbb{N}^*, \sigma)\) \textit{if and only if} \((X, f)\) \textit{is chain transitive}.
Proof sketch

$\mathbb{N}^*$ is chain transitive, and this property is preserved by taking quotients. Therefore every quotient of $\mathbb{N}^*$ is chain transitive.
Proof sketch

\(\mathbb{N}^*\) is chain transitive, and this property is preserved by taking quotients. Therefore every quotient of \(\mathbb{N}^*\) is chain transitive. The more difficult direction of the proof is to show that every chain transitive metrizable dynamical system is a quotient of \(\mathbb{N}^*\).
\( \mathbb{N}^* \) is chain transitive, and this property is preserved by taking quotients. Therefore every quotient of \( \mathbb{N}^* \) is chain transitive. The more difficult direction of the proof is to show that every chain transitive metrizable dynamical system is a quotient of \( \mathbb{N}^* \). To do this, we will actually show

**Proposition**

*If \((X, f)\) is chain transitive and \(X\) metrizable, then there is a metrizable \(Y \supseteq X\), a continuous \(g : Y \to Y\) with \(g \upharpoonright X = f\), and a point \(y \in Y\), such that \(\omega(y) = X\).*
Sketch proof
Sketch proof

\[ X \]
Sketch proof
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\[ X \times (\omega + 1) \]
Let $\{d_n : n \in \mathbb{N}\}$ be a countable dense subset of $X$. 
Let \( \{d_n : n \in \mathbb{N}\} \) be a countable dense subset of \( X \).
In the $n^{th}$ copy of $X$, fix a $\frac{1}{2^n}$-chain from $d_n$ to $d_{n+1}$. 

Sketch proof
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Remove the first point from each of these chains.
Sketch proof

Let $\langle b_n : n \in \mathbb{N} \rangle$ be the natural enumeration of the black points.
Sketch proof

Let $Y$ be $\{b_n : n \in \mathbb{N}\}$ plus the “limit copy” of $X$. Define $g$ on $Y$ by putting $g(b_n) = b_{n+1}$ and $g = f$ on the limit copy.
Sketch proof

This map is continuous, so \((Y, g)\) is a metrizable dynamical system. Furthermore, \(\omega(b_0)\) is the limit copy of \(X\).